

# THE WEIGHT REDUCTION OF MOD $p$ SIEGEL MODULAR FORMS FOR $GS p_4$

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**ABSTRACT.** In this paper we investigate the (classical) weights of mod  $p$  Siegel modular forms of degree 2 toward studying Serre's conjecture for  $GS p_4$ . We first construct various theta operators on the space of such forms a la Katz and define the theta cycles for the specific theta operators. Secondly we study the partial Hasse invariants on each Ekedahl-Oort stratum and their local behaviors. This enables us to obtain a kind of weight reduction theorem for mod  $p$  Siegel modular forms of degree 2 without increasing the level.

## 1. INTRODUCTION

Let  $f$  be an elliptic Hecke eigen cusp form of level  $N$  and weight  $k$  with character  $\varepsilon$ . It is well-known that for each prime  $p$ , one can associate  $f$  with a mod  $p$  Galois representation  $\bar{\rho}_{f,p} : G_{\mathbb{Q}} := \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \text{GL}_2(\bar{\mathbb{F}}_p)$  by Deligne et al. On the other hand, for any odd, irreducible Galois representation  $\bar{\rho} : G_{\mathbb{Q}} \longrightarrow \text{GL}_2(\bar{\mathbb{F}}_p)$ , in a celebrated paper [51], J-P. Serre defined the three invariants  $(k(\bar{\rho}), N(\bar{\rho}), \varepsilon(\bar{\rho}))$  which are called weight, level, and a character in this order. First two invariants are positive integers and the last invariant is a finite character of  $G_{\mathbb{Q}}$  unramified outside  $N(\bar{\rho})$ . In that definition the most difficult one is the weight  $k(\bar{\rho})$  and others are easy to define from  $\bar{\rho}$ . We say such  $\bar{\rho}$  is modular if there exists an elliptic Hecke eigen cusp form  $f$  such that  $\bar{\rho} \sim \bar{\rho}_{f,p}$ . If so there exist another Hecke eigen cusp forms with the same property. So it is important to specify a minimal choice of the weights (also the levels and the characters) among the candidates. Serre conjectured if  $\bar{\rho}$  is modular, then one can find an elliptic Hecke eigen cusp form  $f$  with the weight  $k(\bar{\rho})$  such that  $\bar{\rho} \sim \bar{\rho}_{f,p}$ . This conjecture is called the Serre's weight conjecture (in  $GL_2$ ) and it has been proved by Edixhoven in [15]. After that, the modularity of above  $\bar{\rho}$  was completely proved by Khare-Wintenberger and the Serre's weight conjecture played an essential role there (see [40],[41]). In [15], Edixhoven exploited the theory of mod  $p$  modular forms by Katz (cf. [38]) and theta cycles studied by Jochnowitz [36]. Plugging these into several deep properties of Galois representations, he proved the Serre's weight conjecture. The weight conjecture is now understood and generalized in more general settings in the context of several philosophies as mod  $p$  local Langlands conjecture and how they can lift to a crystalline lift (cf. [32], [10],[20]).

Let  $GS p_4 = GS p_J$  be the symplectic similitude group in  $GL_4$  associated to  $J = \begin{pmatrix} 0_2 & s \\ -s & 0_2 \end{pmatrix}$ ,  $s = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$  with the similitude character  $\nu : GS p_4 \longrightarrow GL_1$ . In this paper, we are concerned with the weight conjecture for  $GS p_4$ . Let  $S_{N,p}$  be the Siegel modular threefold over  $\bar{\mathbb{F}}_p$  with the principal level

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$N \geq 3$ . Let  $F$  be a mod  $p$  Hecke eigen Siegel cusp form of degree 2 with the level  $N$  and the weight  $(k_1, k_2)$ ,  $k_1 \geq k_2 \geq 1$  which is regarded as a global section of an automorphic sheaf on  $S_{N,p}$  (see the next section for the precise definition). Thanks to the works [53],[55],[54],[46],[58],[59], multiplying by the Hasse invariant if necessary, one can associate  $F$  with a mod  $p$  Galois representation  $\bar{\rho}_{F,p} : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_4(\overline{\mathbb{F}}_p)$ . It is easy to see that in fact  $\bar{\rho}_{F,p}$  takes the values in  $\mathrm{GSp}_4(\overline{\mathbb{F}}_p)$ . It also satisfies that  $\nu(\bar{\rho}_{F,p}(c)) = -1$  for any complex conjugation  $c$  and we call such  $\bar{\rho}_{F,p}$  symplectically odd with this property. Conversely one might expect that any symplectically odd, irreducible Galois representation  $\bar{\rho} : G_{\mathbb{Q}} \longrightarrow \mathrm{GSp}_4(\overline{\mathbb{F}}_p)$  is modular, namely, there exists an above mentioned  $F$  such that  $\iota \circ \bar{\rho} \sim \iota \circ \bar{\rho}_{F,p}$  in  $\mathrm{GL}_4(\overline{\mathbb{F}}_p)$  where  $\iota : \mathrm{GSp}_4 \hookrightarrow \mathrm{GL}_4$  is the natural inclusion. The main purpose of this paper is to study a kind of weight reduction theorem for the classical weights of mod  $p$  Siegel modular forms of degree 2. Then we will prove the following theorem:

**Theorem 1.1.** *Suppose  $p \geq 5$ . Let  $\bar{\rho} : G_{\mathbb{Q}} \longrightarrow \mathrm{GSp}_4(\overline{\mathbb{F}}_p)$  be a symplectically odd, continuous representation which is not necessarily irreducible. Assume that  $\bar{\rho}$  comes from a mod  $p$  Hecke eigen Siegel cusp form of degree 2, then there exist an integer  $0 \leq \alpha \leq p-2$  and a mod  $p$  Hecke eigen Siegel cusp form  $F$  of weight  $(k_1, k_2)$ ,  $k_1 \geq k_2 \geq 1$  such that*

- (1)  $p > k_1 - k_2 + 3$  and  $k_2 \leq p^4 + p^2 + 2p + 1$ ,
- (2)  $F$  is not identically zero on the superspecial locus in  $S_{N,p}$ ,
- (3)  $\iota \circ \bar{\rho} \sim \iota \circ (\bar{\chi}_p^\alpha \otimes \bar{\rho}_{F,p})$

where  $\bar{\chi}_p$  stands for the mod  $p$  cyclotomic character of  $G_{\mathbb{Q}}$ .

It will be seen that one can reduce  $k_1 - k_2$  up to modulo 2 as small as possible by using the operator  $\theta_1^{(k_1, k_2)}$  though  $k_2$  would be changed. Therefore we also have the following theorem:

**Theorem 1.2.** *Suppose  $p \geq 5$ . Let  $\bar{\rho} : G_{\mathbb{Q}} \longrightarrow \mathrm{GSp}_4(\overline{\mathbb{F}}_p)$  be a symplectically odd, continuous representation which is not necessarily irreducible. Assume that  $\bar{\rho}$  comes from a mod  $p$  Hecke eigen Siegel cusp form of degree 2 of some weight  $(k'_1, k'_2)$ . Put  $\varepsilon = \frac{1}{2}(1 - (-1)^{k'_1 - k'_2}) \in \{0, 1\}$ . Then for each  $i$  satisfying  $0 \leq i \leq \frac{p-1}{4}$ , there exist an integer  $0 \leq \alpha \leq p-2$  and a mod  $p$  Hecke eigen Siegel cusp form  $F$  of weight  $(k_1, k_2)$ ,  $k_1 \geq k_2 \geq 1$  such that*

- (1)  $k_1 - k_2 = \varepsilon + 2i$  and  $k_2 \leq p^4 + p^2 + 2p + 1$ ,
- (2)  $F$  is not identically zero on the superspecial locus in  $S_{N,p}$ ,
- (3)  $\iota \circ \bar{\rho} \sim \iota \circ (\bar{\chi}_p^\alpha \otimes \bar{\rho}_{F,p})$ .

The (classical) weight of a mod  $p$  Siegel modular form is defined by a pair of integers  $(k_1, k_2)$ ,  $k_1 \geq k_2 \geq 1$  and it is corresponding to the algebraic representation  $\mathrm{Sym}^{k_1 - k_2} \mathrm{St}_2 \otimes \det^{k_2} \mathrm{St}_2$  of  $\mathrm{GL}_2/\overline{\mathbb{F}}_p$ . The above theorem enables us to narrow down the possible weights to a range in the weights as above via mod  $p$  Galois representations up to twists by the mod  $p$  cyclotomic character.

To prove the main theorem we first define several kinds of theta operators and study the non-vanishing of the theta operators by using local moduli around superspecial points in the Siegel modular threefold  $S_{N,p}$  over  $\overline{\mathbb{F}}_p$ . By using these operators and working on the superspecial locus with Ghitza's results ([22],[23]), we will reduce the difference  $k_1 - k_2$  to satisfy  $p > k_1 - k_2 + 3$ . This process can be continued if necessary until  $k_1 - k_2$  becomes to 0 or 1 depending on the parity of the difference of the original weights. Next we will reduce  $k_2$  by making use of the partial Hasse invariants on Ekedahl-Oort strata for  $S_{N,p}$  and their extensions to the Zariski closure of each stratum.

The partial Hasse invariants we use here are defined by Oort in [50] and several people have studied how such invariants extend (see [9],[43],[14]). In this paper we will apply the method of [14] which shows us not only a way to extend but also the local behaviors of the partial Hasse invariants along the boundary. Then we restrict our mod  $p$  Siegel modular forms on each stratum. The bottom stage is the superspecial locus and there one can reduce  $k_2$  to be less than  $p + 1$ . Then we will lift them up to a form on  $S_{N,p}$  keeping the difference  $k_1 - k_2$  but the vanishing of the obstruction for the liftability constraints us so that we have to increase  $k_2$  depending on the weights of partial Hasse invariants.

Our future purpose is definitely to study  $\bar{\rho}_{F,p}|_{G_{\mathbb{Q}_p}}$  where  $G_{\mathbb{Q}_p} := \text{Gal}(\bar{\mathbb{Q}_p}/\mathbb{Q}_p)$ . It should be studied in terms of  $p$ -adic Hodge theory as in [6], [5], [11], [64]. In the reducible case we would be able to apply these results. However if  $\bar{\rho}_{F,p}|_{G_{\mathbb{Q}_p}}$  is irreducible we need to extend the previous result to the higher dimensional case. Further as the readers would have realized, the bound for  $k_2$  in the main theorem is too big and this will cause a trouble to study the integral  $p$ -adic Hodge structure of the representations in question. We would address this program somewhere else.

On the other hand, apart from the main theorems, the theta cycles developed in this paper are apparently different from one in [36], [15] and are of independent interest. It seems also interesting to study a relation to the conjectural Serre weights defined in [33], [57] as irreducible representations over  $\bar{\mathbb{F}_p}$  of  $\text{GSp}_4(\mathbb{F}_p)$ , though the scalar valued case is excluded there where some of them have been confirmed as expected weights by [21].

The second main theorem had been an author's previous main result. However it would not be a suitable way when we compare the classical weights with the local representation  $\bar{\rho}_{F,p}|_{G_{\mathbb{Q}_p}}$  or  $\bar{\rho}_{F,p}|_{G_{I_p}}$ .

We will organize this paper as follows. In Section 2 we recall basics of Siegel modular forms of degree 2. The readers who are familiar with such objects should skip this section. The explicit formulas for Hecke operators in Section 2 will be used in the next section to compute the effect of Hecke eigenvalues under theta operators. Section 3 is devoted to study mod  $p$  Siegel modular forms and define various theta operators a la Katz. In Section 5, 6 we give definition of theta cycles and study their basic properties. In Section 7 we will recall mod  $p$  Galois representation attached to mod  $p$  Siegel modular forms of degree 2 and study elementary properties of those images. Finally, in the appendix we give an explicit form of Pieri's formula for a non-canonical decomposition of the tensor product of two symmetric representations of  $GL_2/\bar{\mathbb{F}_p}$ . This decomposition will be used to construct various theta operators. Under preparing this paper the author realized that Max Flander and Ghitza-McAndrew have studied similar operators for mod  $p$  Siegel modular forms of general degree (see [18], [24]). As the author has been expected, some of results in this paper have been generalized in [24].

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## 2. SIEGEL MODULAR FORMS OF DEGREE 2

In this section we shall discuss the Siegel modular forms in various settings. As basic references, we refer [3] for the classical setting, [8] for the adelic setting, and [29], [55], [52] for the geometric setting. In this section we will work on  $GSp_4 = GSp_{J'}$  defined by  $J' = \begin{pmatrix} 0_2 & I_2 \\ -I_2 & 0_2 \end{pmatrix}$ . It is easy to see that  $GSp_J$  is conjugate to  $GSp_{J'}$  in  $GL_4$  and we can convert everything from one to another.

Let  $\nu$  be the similitude character of  $GSp_4 = GSp_{J'}$  and  $Sp_4$  the kernel of  $\nu$ .

**2.1. Classical Siegel modular forms.** Let us consider the Siegel upper half-plane  $\mathcal{H}_2 = \{Z \in M_2(\mathbb{C}) \mid {}^t Z = Z, \operatorname{Im}(Z) > 0\}$ . For a pair of integers  $\underline{k} = (k_1, k_2)$  such that  $k_1 \geq k_2 \geq 1$ , we define the algebraic representation  $\lambda_{\underline{k}}$  of  $GL_2$  by

$$V_{\underline{k}} = \operatorname{Sym}^{k_1-k_2} \operatorname{St}_2 \otimes \det^{k_2} \operatorname{St}_2,$$

where  $\operatorname{St}_2$  is the standard representation of dimension 2 with the basis  $\{e_1, e_2\}$ . More explicitly, if  $R$  is any ring, then  $V_{\underline{k}}(R) = \bigoplus_{i=0}^{k_1-k_2} R e_1^{k_1-k_2-i} \cdot e_2^i$  and for  $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(R)$ ,  $\lambda_{\underline{k}}(g)$  acts on  $V_{\underline{k}}(R)$  by

$$g \cdot e_1^{k_1-k_2-i} \cdot e_2^i := \det(g)^{k_2} (ae_1 + ce_2)^{k_1-k_2-i} \cdot (be_1 + de_2)^i.$$

We identify  $V_{\underline{k}}(R)$  (resp.  $\lambda_{\underline{k}}(g)$ ) with  $R^{\oplus(k_1-k_2)}$  (resp. the represent matrix of  $\lambda_{\underline{k}}(g)$  with respect to the above basis). We have the action and the automorphy factor  $J$  by

$$(2.1) \quad \gamma Z = (AZ + B)(CZ + D)^{-1}, \quad J(\gamma, Z) = CZ + D \in GL_2(\mathbb{C}),$$

for  $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_4(\mathbb{R})$  and  $Z \in \mathcal{H}_2$ .

For an integer  $N \geq 1$ , we define a principal congruence subgroup  $\Gamma(N)$  to be the group consisting of the elements  $g \in Sp_4(\mathbb{Z})$  such that  $g \equiv 1 \pmod{N}$ .

For a holomorphic  $V_{\underline{k}}(\mathbb{C})$ -valued function  $f$  on  $\mathcal{H}_2$ , the action of  $\gamma \in G(\mathbb{R})^+$  is defined by

$$(2.2) \quad f(Z)|[\gamma]_{\underline{k}} := \lambda_{\underline{k}}(\nu(\gamma)J(\gamma, z)^{-1})f(\gamma Z).$$

For an arithmetic subgroup  $\Gamma$  of  $Sp_4(\mathbb{Q})$  and a finite character  $\chi : \Gamma \rightarrow \mathbb{C}^\times$ , we say that a holomorphic function  $f : \mathcal{H}_2 \rightarrow V_{\underline{k}}(\mathbb{C})$  is a Siegel modular form of weight  $(k_1, k_2)$  with the character  $\chi$  with respect to  $\Gamma$  if it satisfies  $f|[\gamma]_{\underline{k}} = \chi(\gamma)f$  for any  $\gamma \in \Gamma$ . We denote by  $M_{\underline{k}}(\Gamma, \chi)$  the space of such Siegel modular forms. For a Siegel modular form  $f \in M_{\underline{k}}(\Gamma, \chi)$ , the Siegel  $\Phi$ -operator is defined by

$$\Phi(f)(z) := \lim_{t \rightarrow \infty} f\left(\begin{pmatrix} z & 0 \\ 0 & \sqrt{-1}t \end{pmatrix}\right) \text{ for } z \in \mathcal{H}_1$$

where  $\mathcal{H}_1 = \{z \in \mathbb{C} \mid \operatorname{Im}(z) > 0\}$  and we say  $f$  is a cusp form if  $\Phi(f|[\gamma]_{\underline{k}}) = 0$  for any  $\gamma \in Sp_4(\mathbb{Q})$ . We denote by  $S_{\underline{k}}(\Gamma, \chi)$  the space of such cusp forms inside  $M_{\underline{k}}(\Gamma, \chi)$ .

We shall define the Hecke operators on  $M_{\underline{k}}(\Gamma(N))$  (we refer [17]). For any positive integer  $n$  coprime to  $N$ , let

$$\Delta_n(N) := \left\{ g \in M_4(\mathbb{Z}) \cap GSp_4(\mathbb{Q}) \mid g \equiv \begin{pmatrix} I_2 & 0 \\ 0 & \nu(g)I_2 \end{pmatrix} \pmod{N}, \nu(g)^{\pm 1} \in \mathbb{Z}[\frac{1}{n}]^\times \right\}.$$

For  $m \in \Delta_n(N)$ , we introduce the actions of the Hecke operators on  $M_{\underline{k}}(\Gamma(N))$  by

$$(2.3) \quad T_m f(Z) := \nu(m)^{\frac{k_1+k_2}{2}-3} \sum_{\alpha \in \Gamma(N) \setminus \Gamma(N)m\Gamma(N)} f(Z)|[(\nu(m)^{-\frac{1}{2}}\alpha)]_{\underline{k}}$$

and for any positive integer  $n$ , put

$$T(n) := \sum_{m \in \Gamma(N) \setminus \Delta_n(N)} T_m.$$

We call the factor  $\nu(m)^{\frac{k_1+k_2}{2}-3}$  in the formula (2.3) the normalizing factor of  $T_m$  for the weight  $\lambda_{\underline{k}}$ . We also consider the same actions on  $S_{\underline{k}}(\Gamma(N))$ . For  $t_1 = \text{diag}(1, 1, p, p)$ ,  $t_2 = \text{diag}(1, p, p^2, p)$ , put  $T_{i,p} := T_{t_i}$   $i = 1, 2$  and fix  $\tilde{S}_{p,1}, \tilde{S}_{p,p} \in Sp_4(\mathbb{Z})$  so that  $\tilde{S}_{p,1} \equiv \text{diag}(p^{-1}, 1, p, 1) \pmod{N}$  and  $\tilde{S}_{p,p} \equiv \text{diag}(p^{-1}, p^{-1}, p, p) \pmod{N}$ . Put  $S_p := \tilde{S}_{p,p} T_{pI_4} = p^{(k_1+k_2-6)} \tilde{S}_{p,p}$  and note that it commutes with any Hecke operator. Then we see that

$$(2.4) \quad T(p) = T_{1,p}, \quad T_{1,p}^2 - T(p^2) - p^2 S_p = p\{T_{2,p} + (1+p^2)S_p\}.$$

Since the group  $\Gamma(N)$  contains the subgroup consists of  $\begin{pmatrix} I_2 & NS \\ 0 & I_2 \end{pmatrix}, S = {}^t S \in M_2(\mathbb{Z})$ , for a given  $F \in M_{\underline{k}}(\Gamma(N))$ , we have the Fourier expansion

$$(2.5) \quad F(q) = \sum_{T \in \mathcal{S}(\mathbb{Z})_{\geq 0}} A_F(T) q_N^T, \quad q_N^T = e^{\frac{2\pi\sqrt{-1}}{N} \text{tr}(TZ)}$$

where  $\mathcal{S}(\mathbb{Z})_{\geq 0}$  is the subset of  $M_2(\mathbb{Q})$  consisting of all symmetric matrices  $\begin{pmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{pmatrix}$ ,  $a, b, c \in \mathbb{Z}$  which are semi-positive.

In terms of Fourier coefficients, for any  $\mathbb{Z}$ -subalgebra  $R$  of  $\mathbb{C}$  we define

$$M_{\underline{k}}(\Gamma(N), R) := \{F \in M_{\underline{k}}(\Gamma(N)) \mid A_F(T) \in V_{\underline{k}}(R) \text{ for all } T \in \mathcal{S}(\mathbb{Z})_{\geq 0}\}$$

and  $S_{\underline{k}}(\Gamma(N), R)$  as well. Finally for any discrete subgroup  $\Gamma$  of  $Sp_4(\mathbb{Z})$  with finite index and a character  $\chi : \Gamma \rightarrow \mathbb{C}^\times$  so that  $\text{Ker}(\chi)$  contains a principal congruence subgroup  $\Gamma(N')$  for some  $N' > 0$ , put

$$M_{\underline{k}}(\Gamma, \chi, R) = M_{\underline{k}}(\Gamma, \chi) \cap M_{\underline{k}}(\Gamma(N'), R), \quad S_{\underline{k}}(\Gamma, \chi, R) = S_{\underline{k}}(\Gamma, \chi) \cap S_{\underline{k}}(\Gamma(N'), R).$$

Here we take the intersection inside of  $M_{\underline{k}}(\Gamma(N'))$  which includes  $M_{\underline{k}}(\Gamma, \chi)$  and it is as well for  $S_{\underline{k}}(\Gamma, \chi, R)$ . We should remark that the Hecke operators do not preserve  $M_{\underline{k}}(\Gamma, \chi)$  for a general  $\Gamma$  and  $\chi$  (cf. p.465-466 of [47]).

**2.2. A formula for Hecke operators.** The finite group  $Sp_4(\mathbb{Z}/N\mathbb{Z}) \simeq Sp_4(\mathbb{Z})/\Gamma(N)$  acts on  $M_{\underline{k}}(\Gamma(N))$  by  $F \mapsto F|[\tilde{\gamma}]_{\underline{k}}$  if we fix a lift  $\tilde{\gamma}$  of  $\gamma \in Sp_4(\mathbb{Z}/N\mathbb{Z})$ . We denote this action by the same notation  $F|[\gamma]_{\underline{k}}$ . This action does not depend on the choice of any lift of  $\gamma$ . The diagonal subgroup of  $Sp_4(\mathbb{Z}/N\mathbb{Z})$  is isomorphic to  $(\mathbb{Z}/N\mathbb{Z})^\times \times (\mathbb{Z}/N\mathbb{Z})^\times$  by sending  $S_{a,b} := \text{diag}(a^{-1}, b^{-1}, a, b)$  to  $(a, b)$  and it also acts on  $M_{\underline{k}}(\Gamma(N))$  factor through the action of  $Sp_4(\mathbb{Z}/N\mathbb{Z})$ . Then we have the character decomposition

$$(2.6) \quad M_{\underline{k}}(\Gamma(N)) = \bigoplus_{\chi_1, \chi_2 : (\mathbb{Z}/N\mathbb{Z})^\times \rightarrow \mathbb{C}^\times} M_{\underline{k}}(\Gamma(N), \chi_1, \chi_2),$$

where  $M_{\underline{k}}(\Gamma(N), \chi_1, \chi_2) = \{F \in M_{\underline{k}}(\Gamma(N)) \mid F|S_{a,1}]_{\underline{k}} = \chi_1(a)F \text{ and } F|S_{a,a}]_{\underline{k}} = \chi_2(a)F\}$ . It is easy to see that the Hecke operators preserve  $M_{\underline{k}}(\Gamma(N), \chi_1, \chi_2)$  (cf. [47]). We should remark that if the space is non-zero, the weight  $(k_1, k_2)$  has to satisfy the parity condition

$$(2.7) \quad \chi_2(-1) = (-1)^{k_1+k_2}.$$

Throughout this paper we keep to remind this parity condition.

Let  $F(q) = \sum_{T \in \mathcal{S}(\mathbb{Z})_{\geq 0}} A_F(T) q_N^T \in M_{\underline{k}}(\Gamma(N), \chi_1, \chi_2)$  be an eigenform for all  $T(p^i)$ ,  $p \nmid N$ ,  $i \in \mathbb{N}$  with eigenvalues  $\lambda_F(p^i)$ , i.e.,

$$(2.8) \quad T(p^i)F = \lambda_F(p^i)F.$$

Let us study the relation between  $\lambda_F(p^i)$  and  $A_F(T)$ . For a non-negative integer  $\beta$ , let  $R(p^\beta)$  be the set of matrices  $\begin{pmatrix} u_1 & u_2 \\ u_3 & u_4 \end{pmatrix}$  of  $\Gamma^1(N) := \{g \in SL_2(\mathbb{Z}) \mid g \equiv 1_2 \pmod{N}\}$  whose first rows  $(u_1, u_2)$  run over a complete set of representatives modulo the equivalence relation:

$$(u_1, u_2) \sim (u'_1, u'_2) \iff [u_1 : u_2] = [u'_1 : u'_2] \text{ in } \mathbb{P}^1(\mathbb{Z}/p^\beta\mathbb{Z}).$$

Put  $T(p^i)F = \sum_{T \in \mathcal{S}(\mathbb{Z})_{\geq 0}} A_F(p^i; T) q_N^T$ . For simplicity we write  $\rho_j = \text{Sym}^j \text{St}_2$  for  $j \geq 0$  and  $UT^tU = \begin{pmatrix} a_U & \frac{b_U}{2} \\ \frac{b_U}{2} & c_U \end{pmatrix}$  for  $T \in \mathcal{S}(\mathbb{Z})_{\geq 0}$  and  $U \in R(p^\beta)$ . By using Proposition 3.1 of [17] and the calculations done at p.439-440 loc.cit., we have

$$(2.9) \quad \lambda_F(p^i) A_F(T) = A_F(p^i; T) = \sum_{\substack{\alpha+\beta+\gamma=i \\ \alpha, \beta, \gamma \geq 0}} \chi_1(p^\beta) \chi_2(p^\gamma) p^{\beta(k_1-2)+\gamma(k_1+k_2-3)} \times \\ \sum_{\substack{U \in R(p^\beta) \\ a_U \equiv 0 \pmod{p^{\beta+\gamma}} \\ b_U \equiv c_U \equiv 0 \pmod{p^\gamma}}} \rho_{k_1-k_2} \left( \begin{pmatrix} 1 & 0 \\ 0 & p^\beta \end{pmatrix} U \right)^{-1} A_F \left( p^\alpha \begin{pmatrix} a_U p^{-\beta-\gamma} & \frac{b_U p^{-\gamma}}{2} \\ \frac{b_U p^{-\gamma}}{2} & c_U p^{\beta-\gamma} \end{pmatrix} \right).$$

**Remark 2.1.** Fix an isomorphism  $\overline{\mathbb{Q}}_p \simeq \mathbb{C}$  and put  $R = \overline{\mathbb{Z}}_p$ . By the above formula,  $M_{\underline{k}}(\Gamma(N), R)$  is stable under the action of  $T(p^i)$  for any  $i \geq 0$  provided if  $k_2 \geq 2$ . If  $k_2 = 1$ , this would be false.

**2.3. Siegel modular forms with a general weight.** Recall  $\lambda_{\underline{k}} = \lambda_{(k_1, k_2)} = \rho_{k_1-k_2} \otimes \det^{k_2} \text{St}_2$  where  $\rho_{k_1-k_2} = \text{Sym}^{k_1-k_2} \text{St}_2$  and let  $\lambda_{\underline{k}'} = \lambda_{(k'_1, k'_2)}$  be another weight. As in the previous subsection, we can consider the Siegel modular form  $F$  of the weight  $\lambda_{\underline{k}, \underline{k}'} := \lambda_{(k_1, k_2)} \otimes \lambda_{(k'_1, k'_2)}$  with respect to  $\Gamma(N)$ . We denote by  $M_{\lambda_{\underline{k}, \underline{k}'}}(\Gamma(N))$  the space of such Siegel modular forms. We can also define  $M_{\underline{k}, \underline{k}'}(\Gamma(N), \chi_1, \chi_2)$  by the obvious manner. It is known that  $\lambda_{\underline{k}, \underline{k}'}$  decomposes into the irreducible representations as follows:

$$(2.10) \quad \lambda_{\underline{k}, \underline{k}'} = \lambda_{(k_1, k_2)} \otimes \lambda_{(k'_1, k'_2)} \simeq \bigoplus_{j=0}^{\mu} \lambda_{(k_1+k'_1-j, k_2+k'_2+j)}, \quad \mu = \min\{k_1 - k_2, k'_1 - k'_2\}.$$

Notice that clearly the highest weight that is  $(k_1+k_2)+(k'_1+k'_2)$  is preserved under this decomposition. Therefore as (2.3) we may define the Hecke action on  $F$  by

$$(2.11) \quad T_m F(Z) := \nu(m)^{\frac{k_1+k_2+k'_1+k'_2}{2}-3} \sum_{\alpha \in \Gamma(N) \backslash \Gamma(N) m \Gamma(N)} f(Z) |[(\nu(m)^{-\frac{1}{2}} \alpha)_{\underline{k}, \underline{k}'}], \quad m \in \Delta_n(N)$$



where  $F(Z)|[(\nu(m)^{-\frac{1}{2}}\alpha)_{\underline{k},\underline{k}'} = \lambda_{\underline{k},\underline{k}'}(\nu(m)^{-\frac{1}{2}}\alpha)J((\nu(m)^{-\frac{1}{2}}\alpha, z)^{-1})F(\alpha Z)$ .

Let  $F = \sum_{T \in \mathcal{S}(\mathbb{Z})_{\geq 0}} A_F(T)q_N^T \in M_{\underline{k},\underline{k}'}(\Gamma(N), \chi_1, \chi_2)$  be an eigenform for all  $T(p^i)$ ,  $p \nmid N$ ,  $i \in \mathbb{N}$  with eigenvalues  $\lambda_F(p^i)$ . Let  $T(p^i)F = \sum_{T \in \mathcal{S}(\mathbb{Z})_{\geq 0}} A_F(p^i; T)q_N^T$ . As (2.9), we have

$$(2.12) \quad \lambda_F(p^i)A_F(T) = A_F(p^i; T) = \sum_{\substack{\alpha+\beta+\gamma=i \\ \alpha, \beta, \gamma \geq 0}} \chi_1(p^\beta)\chi_2(p^\gamma)p^{\beta(k_1+k'_1-2)+\gamma(k_1+k'_1+k_2+k'_2-3)} \times \\ \sum_{\substack{U \in R(p^\beta) \\ a_U \equiv 0 \pmod{p^{\beta+\gamma}} \\ b_U \equiv c_U \equiv 0 \pmod{p^\gamma}}} (\rho_{k_1-k_2} \otimes \rho_{k'_1-k'_2}) \left( \begin{pmatrix} 1 & 0 \\ 0 & p^\beta \end{pmatrix} U \right)^{-1} A_F \left( p^\alpha \begin{pmatrix} a_U p^{-\beta-\gamma} & \frac{b_U p^{-\gamma}}{2} \\ \frac{b_U p^{-\gamma}}{2} & c_U p^{\beta-\gamma} \end{pmatrix} \right).$$

According to (2.10), we have the decomposition preserving Hecke actions:

$$M_{\underline{k},\underline{k}'}(\Gamma(N), \chi_1, \chi_2) \simeq \bigoplus_{j=0}^{\mu} M_{(k_1+k'_1-j, k_2+k'_2+j)}(\Gamma(N), \chi_1, \chi_2).$$

**2.4. Adelic forms.** In this section we refer to [8] and [53]. Let  $\mathbb{A}$  be the adèle ring of  $\mathbb{Q}$  and  $\mathbb{A}_f = \widehat{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$  the finite adèle of  $\mathbb{Q}$ . For a positive integer  $N$ , let  $K(N)$  be the group consisting of the elements  $g \in GSp_4(\widehat{\mathbb{Z}})$  such that  $g \equiv 1_4 \pmod{N}$ . Then we see that  $\Gamma(N) = Sp_4(\mathbb{Q}) \cap K(N)$  and  $\nu(K(N)) = 1_4 + N\widehat{\mathbb{Z}}$ . Then it follows from the strong approximation theorem for  $Sp_4$  that

$$(2.13) \quad G(\mathbb{A}) = G(\mathbb{Q})G(\mathbb{R})^+ K_H(N) = G(\mathbb{Q})Z_G(\mathbb{R})^+ Sp_4(\mathbb{R})K_H(N)$$

and

$$(2.14) \quad G(\mathbb{A}) = \coprod_{\substack{1 \leq a < N \\ (a, N) = 1}} G(\mathbb{Q})G(\mathbb{R})^+ d_a K(N) = \coprod_{\substack{1 \leq a < N \\ (a, N) = 1}} G(\mathbb{Q})Z_G(\mathbb{R})^+ Sp_4(\mathbb{R})d_a K(N)$$

where  $d_a$  is the diagonal matrix such that  $(d_a)_p = \text{diag}(a, a, 1, 1)$  if  $p|N$ ,  $(d_a)_p = 1_4$  otherwise.

Let  $I := \sqrt{-1}I_2 \in \mathcal{H}_2$  and  $U(2) = \text{Stab}_{Sp_4(\mathbb{R})}(I)$ . For any open compact subgroup  $U$  of  $GSp_4(\widehat{\mathbb{Z}})$ , we let  $\mathcal{A}_{\underline{k}}(U)^\circ$  denote the subspace of functions  $\phi : GSp_4(\mathbb{Q}) \backslash GSp_4(\mathbb{A}) \rightarrow V_{\underline{k}}(\mathbb{C})$  such that

- (1)  $\phi(guu_\infty) = \lambda_{\underline{k}}(J(u_\infty, I)^{-1})\phi(g)$  for all  $g \in G(\mathbb{A})$ ,  $u \in U$ , and  $u_\infty \in U(2)Z_G(\mathbb{R})^+$  where  $Z_G(\mathbb{R})^+ = \mathbb{R}_{>0}I_4$ .
- (2) for  $h \in G(\mathbb{A})$ , the function

$$\phi_h : \mathcal{H}_2 \rightarrow V_{\underline{k}}(\mathbb{C}), \quad \phi_h(Z) = \phi_h(g_\infty I) := \lambda_{\underline{k}}(J(g_\infty, I))\phi(hg_\infty)$$

is holomorphic where  $Z = g_\infty I$ ,  $g_\infty \in Sp_4(\mathbb{R})$  (note that this definition is independent of the choice of  $g_\infty$ ),

- (3) for  $g \in G(\mathbb{A})$ ,  $\int_{N_*(\mathbb{Q}) \backslash N_*(\mathbb{A})} \phi(ng)dn = 0$  for any parabolic subgroup  $* \in \{B, P, Q\}$  and  $dn$  is the Haar measure on  $N_*(\mathbb{Q}) \backslash N_*(\mathbb{A})$ .

We define similarly  $\mathcal{A}_{\underline{k}}(U)$  by omitting the last condition (3).

Let  $\Gamma(N)_a := Sp_4(\mathbb{Q}) \cap d_a^{-1}K(N)d_a$ . Note that  $\Gamma(N)_a = \Gamma(N)$  for each  $a$ . Then we have the isomorphism

$$(2.15) \quad \mathcal{A}_{\underline{k}}(K(N)) \xrightarrow{\sim} \bigoplus_{\substack{1 \leq a < N \\ (a, N) = 1}} M_{\underline{k}}(\Gamma(N)_a), \quad \phi \mapsto (\phi_{d_a}).$$

The inverse of this isomorphism is given as follows: Let  $F = (F_a)$  be an element of RHS which is a system of Siegel modular forms such that  $F_a \in M_{\underline{k}}(\Gamma(N)_a)$  for each  $a$ . For each  $g \in G(\mathbb{A})$ , there exists a unique  $d_a$  such that  $g = rd_a g_\infty k$  with  $r \in G(\mathbb{Q})$ ,  $g_\infty \in Sp_4(\mathbb{R})Z_G(\mathbb{R})^+$ , and  $k \in K(N)$ . Then we define the function

$$\phi_F(g) = \lambda_{\underline{k}}(J(g_\infty, I))^{-1} F_a(g_\infty I).$$

This gives rise to the inverse of the above isomorphism. We also have the isomorphism  $\mathcal{A}_{\underline{k}}(K(N))^\circ \simeq \bigoplus_{\substack{1 \leq a < N \\ (a, N)=1}} S_{\underline{k}}(\Gamma(N)_a)$  as well (cf. [8] for checking the cuspidality).

Now we restrict the isomorphism (2.15) to a subspace, using the character decomposition (2.6). Given two Dirichlet characters  $\chi_i : (\mathbb{Z}/N\mathbb{Z})^\times \rightarrow \mathbb{C}^\times$ ,  $i = 1, 2$ , associate the characters  $\chi'_i : \mathbb{A}_f \rightarrow \mathbb{C}^\times$  via global class field theory.

Define  $\tilde{\chi} : T(\mathbb{A}_f) \rightarrow \mathbb{C}^\times$  by

$$\tilde{\chi}'(\text{diag}(*, *, c, d) = \chi'_1(d^{-1}c)\chi'_2(d).$$

Choose  $F = (F_a)$  from RHS of (2.15) which satisfies  $F[[S_{z,z}]_{\underline{k}}] = (F_a[[S_{z,z}]_{\underline{k}}]) = (\chi_2(z)F_a) = \chi_2(z)F$  and  $F[[S_{z,1}]_{\underline{k}}] = \chi_1(z)F$ . If we write  $g \in G(\mathbb{A})$  as  $g = rz_\infty d_a g_\infty k \in G(\mathbb{A})$  and take  $z_f \in T(\mathbb{A}_f)$ , then we define the automorphic function attached to  $F$  by

$$\phi_F(gz_f) = \lambda_{\underline{k}}(J(g, I))^{-1} F_a(g_\infty I) \tilde{\chi}(z_f).$$

Then this gives rise to the isomorphism of the subspaces

$$\mathcal{A}_{\underline{k}}(K(N), \tilde{\chi}) \xrightarrow{\sim} \bigoplus_{\substack{1 \leq a < N \\ (a, N)=1}} M_{\underline{k}}(\Gamma(N)_a, \chi_1, \chi_2).$$

We now compute the actions of  $\text{diag}(1, 1, -1, -1) \in GSp_4(\mathbb{R})$  on  $\phi_F \in \mathcal{A}_{\underline{k}}(K(N), \tilde{\chi})$  as follows: Let  $h = (\text{diag}(1, 1, -1, -1), I_{\mathbb{A}_f}) = \text{diag}(1, 1, -1, -1)(I_{GSp_4(\mathbb{R})}, (\text{diag}(1, 1, -1, -1))_p) \in G(\mathbb{Q})(GSp_4(\mathbb{R}) \times GSp_4(\mathbb{A}_f))$ , where  $I_{\mathbb{A}_f}$  (resp.  $I_{GSp_4(\mathbb{R})}$ ) is the identity element. Then we have

$$\begin{aligned} \phi_F(gh) &= \phi_F((r \cdot \text{diag}(1, 1, -1, -1))z_\infty d_a g_\infty k \cdot \text{diag}(1, 1, -1, -1))_p) \\ &= \lambda_{\underline{k}}(J(g, I))^{-1} F_a(g_\infty I) \tilde{\chi}(\text{diag}(1, 1, -1, -1))_p) = \chi_2(-1)\phi_F(g), \end{aligned}$$

since we have assumed  $\chi_2(-1) = (-1)^{k_1+k_2}$ . Hence we have

$$(2.16) \quad \phi_F(\text{diag}(1, 1, -1, -1), I_{\mathbb{A}_f}) = (-1)^{k_1+k_2}.$$

**Remark 2.2.** The space  $M_{\underline{k}}(\Gamma(N))$  is embedded into  $\bigoplus_{\substack{1 \leq a < N \\ (a, N)=1}} M_{\underline{k}}(\Gamma(N)_a)$  by  $F \mapsto (F[[\gamma_a]_{\underline{k}}])_a$ . So given a cusp form  $F \in M_{\underline{k}}(\Gamma(N))$ , we obtain  $\phi_F \in \mathcal{A}_{\underline{k}}(K(N))$  which under the isomorphism (2.15), corresponds to  $(F[[\gamma_a]_{\underline{k}}])_a$ , and  $\phi_F$  gives rise to a cuspidal representation  $\pi_F$ . Conversely, given a cuspidal representation  $\pi$  of  $GSp_4/\mathbb{Q}$ , there exists  $N > 0$  and  $\phi \in \mathcal{A}_{\underline{k}}(K(N))$  which spans  $\pi$ . Under the isomorphism (2.15),  $\phi$  corresponds to  $(F_a)_{\substack{1 \leq a < N \\ (a, N)=1}}$ . For any  $a$ , let  $\pi_{F_a}$  be the cuspidal representation associated to  $F_a$ . Then  $\pi$  and  $\pi_{F_a}$  have the same Hecke eigenvalues for  $p \nmid N$ , and hence in the same  $L$ -packet.

We now study the Hecke operators on  $\mathcal{A}_{\underline{k}}(K(N))$  and its relation to classical Hecke operators. Let  $\phi$  be an element of  $\mathcal{A}_{\underline{k}}(K(N))$  and  $F = (F_a)_a$  be the corresponding element of RHS via the



above isomorphism (2.15). For any prime  $p \nmid N$  and  $\alpha \in G(\mathbb{Q}) \cap T(\mathbb{Q}_p)$ , we define the Hecke action with respect to  $\alpha$

$$\tilde{T}_\alpha \phi(g) := \int_{G(\mathbb{A}_f)} ([K(N)_p \alpha K(N)_p] \otimes 1_{K(N)^p}) \phi(gg_f) dg_f$$

where  $dg_f$  is the Haar measure on  $G(\mathbb{A}_f)$  so that  $\text{vol}(K) = 1$ . Here  $K(N)_p$  is the  $p$ -component of  $K(N)$  and  $K(N)^p$  is the subgroup of  $K(N)$  consists of trivial  $p$ -component.

Then by using (2.14), we can easily see that

$$(2.17) \quad T_\alpha F(Z) = \nu(\alpha)^{\frac{k_1+k_2}{2}-3} \tilde{T}_{\alpha^{-1}} \phi(g)$$

where  $g = rz_\infty g_a g_\infty k$  as above and  $Z = g_\infty I$ . From this relation, up to the factor of  $\nu(\alpha)^{\frac{k_1+k_2}{2}-3}$ , the isomorphism (2.15) preserves Hecke eigenforms in both sides. We turn to explain the relation to classical Siegel eigenforms. Let  $G \in S_{\underline{k}}(\Gamma(N))$  be a Siegel cusp form which is a Hecke eigenform. Then it is easy to see that  $(G)_a$  is an eigenform of  $\bigoplus_{\substack{1 \leq a < N \\ (a, N)=1}} S_{\underline{k}}(\Gamma(N)_a)$ . Hence we have the Hecke eigenform of  $\mathcal{A}_{\underline{k}}(K(N))$  corresponding to  $G$ . The above things are easy to generalize to all open compact subgroup  $U \subset G(\widehat{\mathbb{Z}})$ . We omit the details.

The group  $G(\mathbb{A})$  acts on  $\varinjlim_U \mathcal{A}_{\underline{k}}(U)$  (also on  $\varinjlim_U \mathcal{A}_{\underline{k}}(U)^\circ$ ) by right translation:

$$(h \cdot \phi)(g) := \phi(gh), \text{ for } g, h \in G(\mathbb{A}).$$

For an open compact subgroup  $U \subset G(\widehat{\mathbb{Z}})$ , we say  $U$  is of level  $N$  if  $N$  is the minimum positive integer so that  $U$  contains  $K(N)$ . For such  $U$  of level  $N$  and  $\phi \in \mathcal{A}_{\underline{k}}(U)$  which is an eigenform for all  $T_\alpha$ ,  $\alpha \in G(\mathbb{Q}) \cap T(\mathbb{Q}_p)$  and  $p \nmid N$ , we denote by  $\pi_\phi$  the irreducible maximal subquotient of the representation of  $G(\mathbb{A})$  generated by  $g \cdot \phi$ ,  $g \in G(\mathbb{A})$ . Then  $\pi_\phi$  is an automorphic representation in the sense of [8]. Further if  $\phi \in \mathcal{A}_{\underline{k}}(U)^0$ , then we see that  $\pi_\phi$  is a cuspidal automorphic representation.

**2.5. Geometric Siegel modular forms.** We will discuss arithmetic properties of Siegel modular forms by using the modular interpretation of Siegel modular varieties. Henceforth we assume that  $N \geq 3$ .

For any  $\mathbb{Z}[\frac{1}{N}]$ -scheme  $T$ , we consider the triple  $(A, \lambda, \phi)/T$  such that

- (1)  $A$  is an abelian  $T$ -scheme of relative dimension 2,
- (2)  $\lambda : A \xrightarrow{\sim} A^\vee$  is a principal polarization where  $A^\vee$  is the dual abelian scheme of  $A$  (cf. Chapter I of [13]).
- (3)  $\phi$  is an isomorphism  $A[N] \simeq (\mu_{NT})^2 \oplus (\mathbb{Z}/N\mathbb{Z})_T^2$  so that the composition of  $\phi$  and the symplectic pairing on  $(\mu_{NT})^2 \oplus (\mathbb{Z}/N\mathbb{Z})_T^2$  defined by  $J$  is the Weil pairing  $A[N] \times A[N] \rightarrow \mu_N$  defined by  $\lambda$ .

For given two triple  $(A, \lambda, \phi)$  and  $(A', \lambda', \phi')$  we define the equivalence relation as follows. We denote by  $(A, \lambda, \phi) \sim (A', \lambda', \phi')$  if there exists an  $T$ -isomorphism  $f : A \rightarrow A'$  as abelian  $T$ -schemes such that

$$f^\vee \circ \lambda' \circ f = \lambda, \quad f^* \phi = \phi'$$

where  $f^\vee : A'^\vee \rightarrow A^\vee$  is the dual of  $f$ . Then the functor from the category of  $\mathbb{Z}[\frac{1}{N}]$ -schemes to the category of sets defined by

$$Sch_{\mathbb{Z}[\frac{1}{N}]} \rightarrow Sets, \quad T \mapsto \{(A, \lambda, \phi)/T\} / \sim$$

is representable by a  $\mathbb{Z}[\frac{1}{N}]$ -scheme  $S_{K(N)}$  (cf. [13]). It is well known that analytic description of  $S_{K(N)}(\mathbb{C})$  is

$$S_{K(N)}(\mathbb{C}) = G(\mathbb{Q}) \backslash G(\mathbb{A}_f) / Z_G(\mathbb{R})^+ U(2) = \coprod_{\substack{1 \leq a < N \\ (a, N)=1}} \Gamma(N)_a \backslash \mathcal{H}_2 = \coprod_{\substack{1 \leq a < N \\ (a, N)=1}} \Gamma(N) \backslash \mathcal{H}_2$$

which is a smooth quasi-projective 3-fold. Here the smoothness follows from the neatness of  $K(N)$ .

We now turn to define geometric Siegel modular forms by using an arithmetic toroidal compactification of Siegel moduli scheme  $S_{K(N)}$  (cf. [13]). Let  $\overline{S}_{K(N)}$  be an arithmetic toroidal compactification of  $S_{K(N)}$  over  $\mathbb{Z}[\frac{1}{N}]$  and  $\pi : \mathcal{G} \rightarrow \overline{S}_{K(N)}$  be the universal semi-abelian scheme with the zero section  $e : \overline{S}_{K(N)} \rightarrow \mathcal{G}$ . We define the Hodge bundle on  $\overline{S}_{K(N)}$  by

$$(2.18) \quad \mathcal{E} := e^* \Omega_{\mathcal{G}/\overline{S}_{K(N)}}^1$$

which is called “canonical extensions” while  $\mathcal{E}(-\mathcal{C})$  is called “sub-canonical extensions” (cf. Section 4 of [45]). The canonical extensions relate to (geometric modular forms) and the sub-canonical extensions relate to cusp forms. Recall the algebraic representation  $\lambda_{\underline{k}}$  in Section 2.1. For such  $\lambda_{\underline{k}}$ , we associate the automorphic (coherent) sheaf

$$\omega_{\underline{k}} := \text{Sym}^{k_1 - k_2} \mathcal{E} \otimes_{\mathcal{O}_{\overline{S}_{K(N)}}} \det^{k_2} \mathcal{E}.$$

For any  $\mathbb{Z}[\frac{1}{N}]$ -algebra  $R$ , we define the space of geometric Siegel modular forms over  $R$  by

$$M_{\underline{k}}(K(N), R) := H^0(\overline{S}_{K(N)}, \omega_{\underline{k}} \otimes R) = H^0(S_{K(N)}, \omega_{\underline{k}} \otimes R).$$

The last equality follows from Koecher’s principle (cf. [13]).

Let  $D := \overline{S}_{K(N)} - S_{K(N)}$ . Then the space of geometric Siegel cusp forms are defined by

$$S_{\underline{k}}(K(N), R) := H^0(\overline{S}_{K(N)}, \omega_{\underline{k}}(-D) \otimes R).$$

If  $R = \mathbb{C}$  we have

$$M_{\underline{k}}(K(N), \mathbb{C}) = \bigoplus_{\substack{1 \leq a < N \\ (a, N)=1}} M_{\underline{k}}(\Gamma(N)_a)$$

and it is as well for  $S_{\underline{k}}(K(N), \mathbb{C})$ .

Consider Mumford’s semi-abelian scheme

$$(2.19) \quad \mathcal{A}^{\text{Mum}} := \mathbb{G}_m^2 / \left\langle \begin{pmatrix} q_{11} \\ q_{12} \end{pmatrix}, \begin{pmatrix} q_{12} \\ q_{22} \end{pmatrix} \right\rangle_{\mathbb{Z}}$$

as a principal semi-abelian scheme with level  $N$ -structure over  $T_N := \coprod_{\substack{1 \leq a < N \\ (a, N)=1}} \text{Spec } R_N$  where  $R_N := \mathbb{Z}[\frac{1}{N}][q_{12}^{\pm \frac{1}{N}}][[q_{11}^{\frac{1}{N}}, q_{22}^{\frac{1}{N}}]][q_{11}^{-1}, q_{22}^{-1}]$  (cf. p.4-5 of [35]). By universality of  $\mathcal{G}/\overline{S}_{K(N)}$ , there exists a morphism  $\iota : T_N \rightarrow \overline{S}_{K(N)}$  such that

$$\mathcal{A}^{\text{Mum}} \simeq \mathcal{G} \times_{\overline{S}_{K(N)}} T_N.$$

Then  $\iota^* \omega_{\underline{k}}$  is trivialized and it is isomorphic to the constant sheaf  $\bigoplus_{\substack{1 \leq a < N \\ (a, N)=1}} V_{\underline{k}}(R_N \otimes_{\mathbb{Z}[\frac{1}{N}]} R)$ . There is a natural map  $\omega_{\underline{k}} \rightarrow \iota_* \iota^* \omega_{\underline{k}}$  and by taking global sections, we have a map

$$M_{\underline{k}}(K(N), R) \rightarrow \bigoplus_{\substack{1 \leq a < N \\ (a, N)=1}} V_{\underline{k}}(R_N \otimes_{\mathbb{Z}[\frac{1}{N}]} R).$$

The image of  $F \in M_{\underline{k}}(K(N), R)$  under this map can be written as

$$(2.20) \quad F(q) := \left( \sum_{T \in \mathcal{S}(\mathbb{Z})_{\geq 0}} A_{F_a}(T) q_N^T \right)_{\substack{1 \leq a < N \\ (a, N) = 1}}$$

where  $q_N^T = q_{11}^{\frac{a}{N}} q_{12}^{\frac{b}{N}} q_{22}^{\frac{c}{N}}$  for  $T = \begin{pmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{pmatrix}$ . This expression is so called the  $q$ -expansion of  $F$  and  $q$ -expansion principle (cf. [13]) tells us that the above map is injective, hence all coefficients  $A_F(T)$  determines  $F$ . Furthermore it follows from this that for any  $\mathbb{Z}[\frac{1}{N}]$ -subalgebra  $R'$  of  $R$  and  $F \in M_{\underline{k}}(K(N), R)$ ,  $F$  belongs to  $M_{\underline{k}}(K(N), R')$  if and only if  $A_F(T) \in V_{\underline{k}}(R')$  for any  $T \in \mathcal{S}(\mathbb{Z})_{\geq 0}$ .

As in Section 1.1.6 of [52] we can define geometric Hecke operators  $T(p^i)^{\text{geo}}$ ,  $p \nmid N$  on  $M_{\underline{k}}(K(N), R)$  and  $S_{\underline{k}}(K(N), R)$  which coincide with Hecke operators  $T(p^i)$  on classical Siegel modular forms if  $R = \mathbb{C}$  (see (1.1.6.a) of [52]). Then as (2.9) with (2.20), we have the similar formula for the Fourier coefficients of  $T(p^i)^{\text{geo}} F$  in terms of  $A_{F_a}(T)$ . We often identify  $T(p^i)^{\text{geo}}$  with  $T(p^i)$ .

The following theorems due to Lan and Suh ([44],[45]).

**Theorem 2.3.** *The notation being as above. Let  $p \geq 5$  be a prime not dividing  $N$ . Put  $\overline{S}_{N,p} := \overline{S}_{K(N)} \otimes \overline{\mathbb{F}}_p$  and  $\mathcal{C} = \overline{S}_{N,p} - S_{N,p}$ . Assume  $k_2 > 3$  and  $p > 3 + (k_1 - k_2)$ . Then*

$$H^i(\overline{S}_{N,p}, \mathcal{F}) = 0, \quad i = 1, 2$$

for  $\mathcal{F} \in \{\omega_{\underline{k}}, \omega_{\underline{k}}(-\mathcal{C})\}$ .

*Proof.* The claim follows from Theorem 8.13 of [45] (see also Theorem 4.1 of [44]).  $\square$

### 3. $\theta$ -OPERATORS AND $\theta$ -CYCLES

In this section we will define various  $\theta$ -operators on the space of mod  $p$  Siegel modular forms of degree 2 and then study  $\theta$ -cycles which are an analogue of [38],[36] in the case of elliptic modular forms.

**3.1. ordinary locus.** Let us keep the notation in previous section. Let  $S_{N,p}$  be a connected component of  $S_{K(N)} \otimes \overline{\mathbb{F}}_p$ . Let  $\mathcal{A} \xrightarrow{f} S_{N,p}$  be the universal abelian surface with the zero section  $e : S_{N,p} \rightarrow \mathcal{A}$ . As in previous section, the Hodge vector bundle  $\mathcal{E} := e^* \Omega_{\mathcal{A}/S_{N,p}} = f_* \Omega_{\mathcal{A}/S_{N,p}}$  is locally free sheaf on  $S_{N,p}$  of rank 2. Its determinant  $\omega := \det(\mathcal{E})$  is an ample line bundle on  $S_{N,p}$ . These are nothing but base changes to  $S_{N,p}$  of the objects we have defined in Section 2.5. For any scheme  $X$  over  $\overline{\mathbb{F}}_p$ , we denote by  $F_{\text{abs}}$  the absolute Frobenius map on  $X$  which gives rise to the map  $F_{\text{abs}}^* : \mathcal{O}_X \rightarrow \mathcal{O}_X, a \mapsto a^p$ . For any sheaf  $\mathcal{F}$  on  $X$ , put  $\mathcal{F}^{(p)} = \mathcal{F} \otimes_{\mathcal{O}_X, F_{\text{abs}}^*} \mathcal{O}_X$ . As usual let us consider the following commutative diagram:

$$\begin{array}{ccccc} \mathcal{A} & & & & \\ & \searrow F & & \searrow F_{\text{abs}} & \\ & \mathcal{A}^{(p)} & \xrightarrow{\text{proj}} & \mathcal{A} & \\ & \downarrow f^{(p)} & & \downarrow f & \\ S_{N,p} & \xrightarrow{F_{\text{abs}}} & S_{N,p} & & \end{array}$$

where  $F = F_{\mathcal{A}/S_{N,p}}$  stands for the relative Frobenius map where we often drop the subscription from the notation.

By Messing (see [48]) the Hodge to de Rham spectral sequence

$$E_1^{s,t} = R^t f_* \Omega_{\mathcal{A}/S_{N,p}}^s \implies \mathbb{H}_{\text{dR}}^{s+t}(\mathcal{A}/S_{N,p})$$

degenerates at  $E_1$  (this fact is true for any abelian scheme over any scheme). Applying this to  $f$  and  $f^{(p)}$ , we have

$$(3.1) \quad \begin{array}{ccccccc} 0 & \longrightarrow & f_* \Omega_{\mathcal{A}/S_{N,p}}^1 & \xrightarrow{d_1} & \mathbb{H}_{\text{dR}}^1(\mathcal{A}/S_{N,p}) & \xrightarrow{d_2} & R^1 f_* \mathcal{O}_{\mathcal{A}} \longrightarrow 0 \\ & & \uparrow F^* & & \uparrow F^* & & \uparrow F^* \\ 0 & \longrightarrow & f_*^{(p)} \Omega_{\mathcal{A}^{(p)}/S_{N,p}}^1 & \xrightarrow{d_1^{(p)}} & \mathbb{H}_{\text{dR}}^1(\mathcal{A}^{(p)}/S_{N,p}) & \xrightarrow{d_2^{(p)}} & R^1 f_*^{(p)} \mathcal{O}_{\mathcal{A}^{(p)}} \longrightarrow 0 \end{array}$$

where  $F^*$  stands for the pull-back of the relative Frobenius map  $F$ . By working locally on  $S_{N,p}$ , it is easy to see that  $F^*$  is zero on  $f_*^{(p)} \Omega_{\mathcal{A}^{(p)}/S_{N,p}}^1$ . Hence the map  $d_2^{(p)}$  induces a map

$$(3.2) \quad Fr^* : R^1 f_*^{(p)} \mathcal{O}_{\mathcal{A}^{(p)}} \longrightarrow \mathbb{H}_{\text{dR}}^1(\mathcal{A}^{(p)}/S_{N,p})$$

and also a map

$$(3.3) \quad d_2 \circ Fr^* : R^1 f_*^{(p)} \mathcal{O}_{\mathcal{A}^{(p)}} \longrightarrow R^1 f_* \mathcal{O}_{\mathcal{A}}$$

This gives rise to a map

$$(3.4) \quad \omega^{-p} \longrightarrow \omega^{-1}.$$

Hence by Serre duality, one has a non-zero global section  $H_{p-1}$  on  $S_{N,p}$  of the sheaf  $\text{Hom}(\omega^{-p}, \omega^{-1}) = \omega^{p-1}$ . The section  $H_{p-1}$  is called the Hasse invariant and regarded as a mod  $p$  Segel modular form of weight  $p-1$  with level one.

We denote by  $S_{N,p}^h$  the locus so that the Hasse invariant  $H_{p-1}$  is non-zero. Then the following is known-well.

**Theorem 3.1.** *The locus  $S_{N,p}^h$  is a maximal subscheme so that the Frobenius map (3.2) gives a splitting of the exact sequence (3.7).*

Before proving this theorem, let us recall the de Rham pairing. Since  $\mathcal{A}$  is endowed with a principal polarization  $\lambda : \mathcal{A} \xrightarrow{\sim} \mathcal{A}^\vee$ , there is a canonical isomorphism

$$(3.5) \quad \mathbb{H}_{\text{dR}}^1(\mathcal{A}/S_{N,p}) \simeq \mathbb{H}_{\text{dR}}^1(\mathcal{A}^\vee/S_{N,p}) \simeq \mathbb{H}_{\text{dR}}^1(\mathcal{A}/S_{N,p})^\vee.$$

This induces a canonical alternating pairing

$$(3.6) \quad \langle \cdot, \cdot \rangle_{\text{dR}} : \mathbb{H}_{\text{dR}}^1(\mathcal{A}/S_{N,p}) \times \mathbb{H}_{\text{dR}}^1(\mathcal{A}/S_{N,p}) \simeq \mathbb{H}_{\text{dR}}^1(\mathcal{A}/S_{N,p}) \times \mathbb{H}_{\text{dR}}^1(\mathcal{A}/S_{N,p})^\vee \longrightarrow \mathcal{O}_{S_{N,p}}$$

Let us consider

$$(3.7) \quad \begin{array}{ccccccc} 0 & \longrightarrow & f_* \Omega_{\mathcal{A}/S_{N,p}}^1 & \xrightarrow{d_1} & \mathbb{H}_{\text{dR}}^1(\mathcal{A}/S_{N,p}) & \xrightarrow{d_2} & R^1 f_* \mathcal{O}_{\mathcal{A}} \longrightarrow 0 \\ & & \downarrow & & \downarrow \text{using } \langle \cdot, \cdot \rangle_{\text{dR}} \simeq & & \downarrow \\ 0 & \longleftarrow & (f_* \Omega_{\mathcal{A}/S_{N,p}}^1)^\vee & \xleftarrow{d_1^\vee} & \mathbb{H}_{\text{dR}}^1(\mathcal{A}/S_{N,p})^\vee & \xleftarrow{d_2^\vee} & (R^1 f_* \mathcal{O}_{\mathcal{A}})^\vee \longleftarrow 0. \end{array}$$

Then by chasing diagram, the left vertical map is zero, namely for any local sections  $\omega_1, \omega_2 \in f_*\Omega^1_{\mathcal{A}/S_{N,p}}$ , one has  $\langle \omega_1, \omega_2 \rangle_{\text{dR}} = 0$ . Then one has a canonical isomorphism

$$(3.8) \quad R^1 f_* \mathcal{O}_{\mathcal{A}} \xleftarrow{\sim d_2} \mathbb{H}_{\text{dR}}^1(\mathcal{A}/S_{N,p})/f_*\Omega^1_{\mathcal{A}/S_{N,p}} \xrightarrow{\sim} \mathbb{H}_{\text{dR}}^1(\mathcal{A}/S_{N,p})/(R^1 f_* \mathcal{O}_{\mathcal{A}})^\vee \xrightarrow{\sim d_1^\vee} (f_*\Omega^1_{\mathcal{A}/S_{N,p}})^\vee.$$

This gives rise to a canonical non-degenerate pairing

$$(3.9) \quad \langle \cdot, \cdot \rangle_{\text{dual}} : f_*\Omega^1_{\mathcal{A}/S_{N,p}} \times R^1 f_* \mathcal{O}_{\mathcal{A}} \longrightarrow \mathcal{O}_{S_{N,p}}.$$

For any local section  $\omega_1 \in f_*\Omega^1_{\mathcal{A}/S_{N,p}}$  and a lift  $\tilde{\eta}_1 \in \mathbb{H}_{\text{dR}}^1(\mathcal{A}/S_{N,p})$  of its dual  $\eta_1 \in R^1 f_* \mathcal{O}_{\mathcal{A}}$  via Serre duality (namely  $\langle \omega_1, \eta_1 \rangle_{\text{dual}} = 1$ ). Then we have  $\langle \omega_1, \tilde{\eta}_1 \rangle_{\text{dR}} = \langle \omega_1, \eta_1 \rangle_{\text{dual}} = 1$ .

We now give a proof of Theorem 3.1.

*Proof.* Take a local basis  $\omega_1, \omega_2 \in f_*\Omega^1_{\mathcal{A}/S_{N,p}}$  and the corresponding basis  $\eta_1, \eta_2 \in R^1 f_* \mathcal{O}_{\mathcal{A}}$  via Serre duality. Fix lifts  $\tilde{\eta}_i, i = 1, 2$  of  $\eta_i$  to  $\mathbb{H}_{\text{dR}}^1(\mathcal{A}/S_{N,p})$ . Let us denote by  $\eta_i^{(p)}$  the image of  $\eta_i \otimes 1$  under the Frobenius semilinear map

$$(3.10) \quad F_{\text{abs}}^* \mathbb{H}_{\text{dR}}^1(\mathcal{A}^{(p)}/S_{N,p}) = \mathbb{H}_{\text{dR}}^1(\mathcal{A}/S_{N,p})^{(p)} \longrightarrow \mathbb{H}_{\text{dR}}^1(\mathcal{A}^{(p)}/S_{N,p}).$$

Clearly  $\eta_i^{(p)} \in R^1 f_* \mathcal{O}_{\mathcal{A}^{(p)}}$  for  $i = 1, 2$  and they make up a basis. Then one has

$$(3.11) \quad Fr^*(\eta_1^{(p)}, \eta_2^{(p)}) = (\omega_1, \omega_2, \tilde{\eta}_1, \tilde{\eta}_2) \begin{pmatrix} B \\ A \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}, \quad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in M_2(\mathcal{O}_{S_{N,p}}).$$

The matrix  $A$  also satisfies  $d_2 \circ Fr^*(\eta_1^{(p)}, \eta_2^{(p)}) = (\eta_1, \eta_2)A$  by functoriality of the Hodge filtration (see (3.2) for  $d_2 \circ Fr^*$ ). The claim is equivalent to requiring that  $\text{Im}(Fr^*)$  is of rank 2 and it occurs exactly when  $H_{p-1} = \det({}^t A) = \det(A) \neq 0$ .  $\square$

**3.2. The Gauss-Manin connection.** In this subsection we will recall the basic facts for the Gauss-Manin connection and the Kodaira-Spencer map in our setting. We will try to make everything as explicit as possible. We refer [39],[42] for basis references.

For our smooth family  $\mathcal{A}/S_{N,p}$ , one can associate an integral connection which is so called the Gauss-Manin connection:

$$\nabla : \mathbb{H}_{\text{dR}}^1(\mathcal{A}/S_{N,p}) \longrightarrow \mathbb{H}_{\text{dR}}^1(\mathcal{A}/S_{N,p}) \otimes_{\mathcal{O}_{S_{N,p}}} \Omega^1_{S_{N,p}}.$$

It is well-known that, for example, see [30] that  $\text{Hom}_{\mathcal{O}_{S_{N,p}}}(\Omega^1_{S_{N,p}}, \mathcal{O}_{S_{N,p}}) = (\Omega^1_{S_{N,p}})^\vee$  is naturally identified with the sheaf of derivatives on  $\mathcal{O}_{S_{N,p}}$  which is denoted by  $\text{Der}(\mathcal{O}_{S_{N,p}})$ . For a given local section  $D$  of  $\text{Der}(\mathcal{O}_{S_{N,p}})$  one has

$$(3.12) \quad \begin{array}{ccc} \mathbb{H}_{\text{dR}}^1(\mathcal{A}/S_{N,p}) & \xrightarrow{\nabla} & \mathbb{H}_{\text{dR}}^1(\mathcal{A}/S_{N,p}) \otimes_{\mathcal{O}_{S_{N,p}}} \Omega^1_{S_{N,p}} \\ & \searrow \nabla(D) & \downarrow 1 \otimes D \\ & & \mathbb{H}_{\text{dR}}^1(\mathcal{A}/S_{N,p}). \end{array}$$

Since  $\nabla$  is integrable, for any  $D_1, D_2 \in \text{Der}(\mathcal{O}_{S_{N,p}})$  satisfying  $D_1 D_2 = D_2 D_1$ , one has  $\nabla(D_1) \nabla(D_2) = \nabla(D_2) \nabla(D_1)$  on  $\mathbb{H}_{\text{dR}}^1(\mathcal{A}/S_{N,p})$ .

Choose a local coordinate  $x_{11}, x_{12}, x_{22}$  of  $S_{N,p}$ , then locally  $\Omega_{S_{N,p}}^1 = \langle dx_{11}, dx_{12}, dx_{22} \rangle_{\mathcal{O}_{S_{N,p}}}$ . Take the dual local basis  $\frac{\partial}{\partial x_{11}}, \frac{\partial}{\partial x_{12}}, \frac{\partial}{\partial x_{22}}$  of  $Der(\mathcal{O}_{S_{N,p}})$ . It follows from this that

$$(3.13) \quad \nabla(m) = \sum_{1 \leq i \leq j \leq 2} \nabla(\frac{\partial}{\partial x_{ij}})(m) dx_{ij} \text{ for } m \in \mathbb{H}_{\text{dR}}^1(\mathcal{A}/S_{N,p}).$$

One would use another basis  $d_{11}, d_{12}, d_{22}$  of  $\Omega_{S_{N,p}}^1$  and their dual basis  $D_{11}, D_{12}, D_{22}$  of  $Der(\mathcal{O}_{S_{N,p}})$ . Then one would see locally

$$\nabla(m) = \sum_{1 \leq i \leq j \leq 2} \nabla(D_{ij})(m) d_{ij}, \quad m \in \mathbb{H}_{\text{dR}}^1(\mathcal{A}/S_{N,p}).$$

However the commutativity of  $D_{ij}$ 's does not necessarily hold in general.

Recall the Kodaira-Spencer isomorphism (as  $O_{S_{N,p}}$ -modules)

$$(3.14) \quad KS : \mathrm{Sym}^2 \mathcal{E} \xrightarrow{\sim} \Omega_{S_{N,p}}^1$$

which has the property that for any local basis  $\omega_1, \omega_2$  of  $\mathcal{E}$ ,

$$(3.15) \quad \left\{ \begin{array}{ll} KS(\omega_1^{\otimes 2}) &= \langle \omega_1, \nabla \omega_1 \rangle_{\text{dR}}, \\ KS(\frac{1}{2}(\omega_1 \otimes \omega_2 + \omega_2 \otimes \omega_1)) &= \langle \omega_1, \nabla \omega_2 \rangle_{\text{dR}}, \\ KS(\omega_1^{\otimes 2}) &= \langle \omega_2, \nabla \omega_2 \rangle_{\text{dR}}. \end{array} \right.$$

Let us explain how we gain (3.14) and (3.15). For each local section  $D$  of  $Der(\mathcal{O}_{S_{N,p}})$ , one can associate a  $\mathcal{O}_{S_{N,p}}$ -module homomorphism  $\psi_D :$

$$\begin{array}{ccccccc}
 f_*\Omega_{\mathcal{A}/S_{N,p}}^1 & \xrightarrow{\nabla(D)} & \mathbb{H}_{\mathrm{dR}}^1(\mathcal{A}/S_{N,p}) & \xrightarrow{(3.7) \simeq} & \mathbb{H}_{\mathrm{dR}}^1(\mathcal{A}/S_{N,p})^\vee & \xrightarrow{d_1^\vee} & (f_*\Omega_{\mathcal{A}/S_{N,p}}^1)^\vee \\
 & & & & & & \downarrow (3.8) \simeq \\
 & & & \searrow \psi_D & & & R^1 f_* \mathcal{O}_{\mathcal{A}}.
 \end{array}$$

Thus we have

$$(3.16) \quad \text{Der}(\mathcal{O}_{S_{N,p}}) \longrightarrow \text{Hom}_{\mathcal{O}_{S_{N,p}}}(f_*\Omega_{\mathcal{A}/S_{N,p}}^1, R^1f_*\mathcal{O}_{\mathcal{A}}), D \mapsto \psi_D.$$

We now study the image of this map. Let us fix a local basis  $\omega_1, \omega_2$  of  $\mathcal{E} = f_*\Omega_{\mathcal{A}/S_{N,p}}^1$  and take the local basis  $\eta_1, \eta_2$  of  $R^1f_*\mathcal{O}_{\mathcal{A}}$  via the duality (3.8). We also fix lifts  $\tilde{\eta}_i$ ,  $i = 1, 2$  of  $\eta_i$  to  $\mathbb{H}_{\text{dR}}^1(\mathcal{A}/S_{N,p})$ . For  $1 \leq i, j \leq 2$ , we denote by  $t_{ij}$  the morphism sending  $\omega_i$  to  $\eta_j$  and  $\omega_{i'}, i' \neq i$  to 0. Then  $\{t_{ij}\}_{1 \leq i, j \leq 2}$  makes up a local basis of the RHS of (3.16). On the other hand, if we write

$$(3.17) \quad (\nabla(D)\omega_1, \nabla(D)\omega_2) = (\omega_1, \omega_2, \tilde{\eta}_1, \tilde{\eta}_2) \begin{pmatrix} * \\ A_D \end{pmatrix}, \quad A_D = \begin{pmatrix} a_{11}^D & a_{12}^D \\ a_{21}^D & a_{22}^D \end{pmatrix} \in M_2(\mathcal{O}_{S_{N,p}}),$$

then by Leibniz rule and using  $\langle \omega_1, \omega_2 \rangle_{\text{dR}} = 0$ ,

$$\begin{aligned} 0 &= \nabla(D)\langle\omega_1, \omega_2\rangle_{\text{dR}} = \langle\nabla(D)\omega_1, \omega_2\rangle_{\text{dR}} + \langle\omega_1, \nabla(D)\omega_2\rangle_{\text{dR}} \\ &= -a_{21}^D + a_{12}^D. \end{aligned}$$

It follows from this that  $\psi_D(\omega_i) = \sum_{j=1,2} a_{ij}^D \eta_j$  and therefore we have

$$\psi_D = a_{11}^D t_{11} + a_{12}^D (t_{12} + t_{21}) + a_{22}^D t_{22}.$$



As seen before, if we work locally on  $S_{N,p}$  and for  $1 \leq k \leq l \leq 2$ , put  $a_{ij}^{(kl)} := a_{ij}^D$  for the dual basis  $D_{kl}$  of  $d_{kl}$ , then

$$(3.18) \quad \begin{aligned} \langle \omega_1, \nabla \omega_1 \rangle_{\text{dR}} &= \sum_{1 \leq k \leq l \leq 2} a_{11}^{(kl)} d_{kl}, \\ \langle \omega_1, \nabla \omega_2 \rangle_{\text{dR}} &= \sum_{1 \leq k \leq l \leq 2} a_{12}^{(kl)} d_{kl} = \langle \omega_2, \nabla \omega_1 \rangle_{\text{dR}}, \\ \langle \omega_2, \nabla \omega_2 \rangle_{\text{dR}} &= \sum_{1 \leq k \leq l \leq 2} a_{22}^{(kl)} d_{kl}. \end{aligned}$$

The images of  $t_{11}, (t_{12} + t_{21}), t_{22}$  under a canonical identification

$$(3.19) \quad \begin{aligned} \text{Hom}_{\mathcal{O}_{S_{N,p}}} (f_* \Omega_{\mathcal{A}/S_{N,p}}^1, R^1 f_* \mathcal{O}_{\mathcal{A}}) &= (f_* \Omega_{\mathcal{A}/S_{N,p}}^1)^\vee \otimes_{\mathcal{O}_{S_{N,p}}} R^1 f_* \mathcal{O}_{\mathcal{A}} \\ &\xrightarrow{(3.8)} (f_* \Omega_{\mathcal{A}/S_{N,p}}^1)^\vee \otimes_{\mathcal{O}_{S_{N,p}}} (f_* \Omega_{\mathcal{A}/S_{N,p}}^1)^\vee = \mathcal{E}^\vee \otimes_{\mathcal{O}_{S_{N,p}}} \mathcal{E}^\vee \end{aligned}$$

are given by  $\eta_1^\vee \otimes \eta_1^\vee, (\eta_1^\vee \otimes \eta_2^\vee + \eta_2^\vee \otimes \eta_1^\vee), \eta_2^\vee \otimes \eta_2^\vee$  respectively where  $\eta_i^\vee$  stands for the dual element corresponding to  $\eta_i$  via (3.8). Let us introduce the formal symbol  $(\text{Sym}^2 \mathcal{E})^*$  to be the  $\mathcal{O}_{S_{N,p}}$ -submodule of  $\mathcal{E} \otimes_{\mathcal{O}_{S_{N,p}}} \mathcal{E}$  generated by  $\eta_1^\vee \otimes \eta_1^\vee, (\eta_1^\vee \otimes \eta_2^\vee + \eta_2^\vee \otimes \eta_1^\vee), \eta_2^\vee \otimes \eta_2^\vee$ . Composing (3.16), (3.19) and taking dual, one has the Kodaira-Spencer map

$$\text{Sym}^2 \mathcal{E} \simeq ((\text{Sym}^2 \mathcal{E})^*)^\vee \longrightarrow (\text{Der}(\mathcal{O}_{S_{N,p}}))^\vee = ((\Omega_{S_{N,p}}^1)^\vee)^\vee = \Omega_{S_{N,p}}^1.$$

Since the formation is compatible with base change, by working over  $\mathbb{C}$  (see [29]), this map gives an isomorphism (see also Section 9 in Chapter III of [13]).

Then it follows from this that

$$\begin{aligned} (\eta_1^\vee \otimes \eta_1^\vee)^\vee &\mapsto \langle \omega_1, \nabla \omega_1 \rangle_{\text{dR}} \\ (\eta_1^\vee \otimes \eta_2^\vee + \eta_2^\vee \otimes \eta_1^\vee)^\vee &\mapsto \langle \omega_1, \nabla \omega_2 \rangle_{\text{dR}} + \langle \omega_2, \nabla \omega_1 \rangle_{\text{dR}} = 2 \langle \omega_1, \nabla \omega_2 \rangle_{\text{dR}} \cdot \\ (\eta_2^\vee \otimes \eta_2^\vee)^\vee &\mapsto \langle \omega_2, \nabla \omega_2 \rangle_{\text{dR}}. \end{aligned}$$

This explains exactly how we obtained the Kodaira-Spencer map (3.14).

**3.3. Small  $\theta$ -operators and a big  $\Theta$  operator.** In this subsection we will introduce the “small”  $\theta$ -operators as in [38]. Contrary to the case  $GL_2$ , there are various kinds of theta operators which will be denoted by  $\theta, \theta_1, \theta_2, \theta_3$  and applying  $\theta, \theta_1$  we will also define the “big”  $\Theta$  operator which is a characteristic  $p$  geometric counter part of Bocherer-Nagaoka’s operator in the classical case (see [7]). Throughout this section we always assume  $p \geq 5$ . Let us keep the notation in Section (3.2).

Let  $\omega_1, \omega_2$  be a local basis of  $\mathcal{E}$ . By Kodaira-Spencer isomorphism,  $\langle \omega_i, \nabla \omega_j \rangle_{\text{dR}}, 1 \leq i \leq j \leq 2$  make up a local basis of  $\Omega_{S_{N,p}}^1$ . We denote by  $D_{ij}$  the dual of  $\langle \omega_i, \nabla \omega_j \rangle_{\text{dR}}$ . Then by (3.18) we have

$$(3.20) \quad \begin{aligned} \langle \omega_1, \nabla (D_{11}) \omega_1 \rangle_{\text{dR}} &= 1, \quad \langle \omega_1, \nabla (D_{12}) \omega_1 \rangle_{\text{dR}} = 0, \quad \langle \omega_1, \nabla (D_{22}) \omega_1 \rangle_{\text{dR}} = 0 \\ \langle \omega_1, \nabla (D_{11}) \omega_2 \rangle_{\text{dR}} &= 0, \quad \langle \omega_1, \nabla (D_{12}) \omega_2 \rangle_{\text{dR}} = 1, \quad \langle \omega_1, \nabla (D_{22}) \omega_2 \rangle_{\text{dR}} = 0 \\ \langle \omega_2, \nabla (D_{11}) \omega_2 \rangle_{\text{dR}} &= 0, \quad \langle \omega_2, \nabla (D_{12}) \omega_2 \rangle_{\text{dR}} = 0, \quad \langle \omega_2, \nabla (D_{22}) \omega_2 \rangle_{\text{dR}} = 1. \end{aligned}$$

**Lemma 3.2.** *Keep the notation above. Then the following facts hold.*

- (1)  $\nabla (D_{11}) \omega_2 = \nabla (D_{22}) \omega_1 = 0$ ,
- (2)  $\nabla (D_{11}) \omega_1 = \nabla (D_{12}) \omega_2$  and  $\nabla (D_{22}) \omega_2 = \nabla (D_{12}) \omega_1$ ,
- (3) *The elements  $\omega_1, \omega_2, \nabla (D_{11}) \omega_1, \nabla (D_{22}) \omega_2$  make up a local basis of  $\mathbb{H}_{\text{dR}}^1(\mathcal{A}_{\mathcal{X}}/\mathcal{X})$ .*

*Proof.* We first show (1). Applying (3.17) for  $D_{11}$ , one has  $A_{D_{11}} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ . Hence  $\langle \nabla(D_{11})\omega_2, \tilde{\eta}_i \rangle_{\text{dR}} = 0$  for  $i = 1, 2$ . Further by (3.20),  $\langle \nabla(D_{11})\omega_2, \omega_i \rangle_{\text{dR}} = 0$  for  $i = 1, 2$ . Then it follows from non-degeneracy of the de Rham pairing that  $\nabla(D_{11})\omega_2 = 0$ . Similarly, one has  $\nabla(D_{22})\omega_1 = 0$ .

Next we prove (2). By (3.20), clearly  $\langle \nabla(D_{11})\omega_1 - \nabla(D_{12})\omega_2, \omega_i \rangle_{\text{dR}} = 0$  for  $i = 1, 2$ . Applying (3.17) for  $D_{12}, D_{22}$ , one has  $A_{D_{12}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ ,  $A_{D_{22}} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ . Thus we see that

$$\langle \nabla(D_{11})\omega_1 - \nabla(D_{12})\omega_2, \tilde{\eta}_1 \rangle_{\text{dR}} = 1 - 1 = 0, \quad \langle \nabla(D_{11})\omega_1 - \nabla(D_{12})\omega_2, \tilde{\eta}_2 \rangle_{\text{dR}} = 0 - 0 = 0.$$

The second equality is the same as above.

For the last claim, let us assume  $a_1\omega_1 + a_2\omega_2 + a_3\nabla(D_{11})\omega_1 + a_4\nabla(D_{22})\omega_2 = 0, a_i \in \mathcal{O}_{S_{N,p}}$ . First we apply  $\langle *, \omega_i \rangle_{\text{dR}}, i = 1, 2$  and next do  $\langle *, \tilde{\eta}_i \rangle_{\text{dR}}, i = 1, 2$ . Then one has  $a_1 = a_2 = a_3 = a_4 = 0$ .  $\square$

By using the basis of Lemma 3.2-(3), as (3.11) we consider

$$(3.21) \quad F^*((\nabla(D_{11})\omega_1)^{(p)}, (\nabla(D_{22})\omega_2)^{(p)}) = (\omega_1, \omega_2, \nabla(D_{11})\omega_1, \nabla(D_{22})\omega_2) \begin{pmatrix} B \\ A \end{pmatrix},$$

$$B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}, \quad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in M_2(\mathcal{O}_{S_{N,p}}).$$

Here we use  $F^*$  not  $Fr^*$ . As (3.17), let us fix lifts  $\phi_i, i = 1, 2$  of  $\eta_i$  (here we use the symbol  $\phi_i$  instead of  $\tilde{\eta}_i$ ). Since  $\langle \omega_i, \phi_i \rangle_{\text{dR}} = 1, i = 1, 2$  and (3.20), we see that  $A$  is the Hasse-Matrix with respect to  $\eta_1, \eta_2$  and we have

$$(3.22) \quad (w_1, w_2)B + (\nabla(D_{11})\omega_1, \nabla(D_{22})\omega_2)A = (\phi_1, \phi_2)A.$$

Put

$$\nabla_{11} := \nabla(D_{11}), \quad \nabla_{12} := \nabla(D_{12}), \quad \nabla_{22} := \nabla(D_{22})$$

for simplicity. As seen before, these are not necessarily commutative even though  $\nabla$  is integrable.

The relation between  $A$  and  $B$  is revealed by the following lemma:

**Lemma 3.3.** *Let  $S$  be a scheme over  $\overline{\mathbb{F}}_p$  and let  $f : \mathcal{A} \rightarrow S$  be an abelian scheme. Let  $\nabla^{(p)}$  be the Gauss-Manin connection associated to  $f^{(p)} : \mathcal{A}^{(p)} \rightarrow S$ . Then for any local section  $h$  of  $\mathbb{H}_{\text{dR}}^1(\mathcal{A}/S)$  and any  $D \in \text{Der}(\mathcal{O}_S)$ , we have*

$$\nabla(D)(F^*(h^{(p)})) = 0.$$

*Proof.* By the compatibility of the Gauss-Manin connection with the relative Frobenius map  $F$  and base change  $F_{\text{abs}}$ , one has the following commutative diagram:

$$\begin{array}{ccccc} \mathbb{H}_{\text{dR}}^1(\mathcal{A}/S)^{(p)} & \xrightarrow{h \otimes 1 \mapsto h^{(p)}} & \mathbb{H}_{\text{dR}}^1(\mathcal{A}^{(p)}/S) & \xrightarrow{F^*} & \mathbb{H}_{\text{dR}}^1(\mathcal{A}/S) \\ \downarrow F_{\text{abs}}^* \nabla^{(p)} & & \downarrow \nabla^{(p)} & & \downarrow \nabla \\ \mathbb{H}_{\text{dR}}^1(\mathcal{A}/S)^{(p)} \otimes_{\mathcal{O}_{S^{(p)}}} \Omega_{S^{(p)}}^1 & \xrightarrow{\alpha} & \mathbb{H}_{\text{dR}}^1(\mathcal{A}^{(p)}/S) \otimes_{\mathcal{O}_S} \Omega_S^1 & \xrightarrow{F^*} & \mathbb{H}_{\text{dR}}^1(\mathcal{A}/S) \otimes_{\mathcal{O}_S} \Omega_S^1 \\ \downarrow 1 \otimes D^{(p)} & & \downarrow 1 \otimes D & & \downarrow 1 \otimes D \\ \mathbb{H}_{\text{dR}}^1(\mathcal{A}/S)^{(p)} & \longrightarrow & \mathbb{H}_{\text{dR}}^1(\mathcal{A}^{(p)}/S) & \xrightarrow{F^*} & \mathbb{H}_{\text{dR}}^1(\mathcal{A}/S) \end{array}$$

where three left horizontal arrows are defined as (3.10). Let us observe the left middle horizontal arrow (which has been denoted by  $\alpha$ ). Since  $\Omega_{S^{(p)}}^1 \rightarrow \Omega_S^1$  is locally given by  $df \mapsto df^p = p f^{p-1} df = 0$ , it vanishes, so does the map  $\alpha$ . The element  $\nabla(D)(F^*(h^{(p)}))$  has been factoring through  $\alpha$ . Hence this gives us the claim.  $\square$

**Proposition 3.4.** *Assume that  $\nabla_{ij}$ 's are commutative each other. Then the following relations for  $A$  and  $B$  hold:*

(1)

$$\begin{aligned} \nabla_{11}(a_{11}) &= -b_{11} & \nabla_{12}(a_{11}) &= -b_{21} & \nabla_{22}(a_{11}) &= 0 \\ \nabla_{11}(a_{12}) &= -b_{12} & \nabla_{12}(a_{12}) &= -b_{22} & \nabla_{22}(a_{12}) &= 0 \\ \nabla_{11}(a_{21}) &= 0 & \nabla_{12}(a_{21}) &= -b_{11} & \nabla_{22}(a_{21}) &= -b_{21} \\ \nabla_{11}(a_{22}) &= 0 & \nabla_{12}(a_{22}) &= -b_{12} & \nabla_{22}(a_{22}) &= -b_{22}, \end{aligned}$$

(2) for any  $1 \leq i, j \leq 2$  and  $1 \leq k < l \leq 2$ ,  $\nabla_{kl}(b_{ij}) = 0$ .

*Proof.* By Leibniz rule and Lemma 3.3, for  $1 \leq k < l \leq 2$ ,  $1 \leq i, j \leq 2$ ,

$$\begin{aligned} \nabla_{kl}\langle F^*((\nabla_{ii}\omega_i)^{(p)}), \omega_j \rangle_{\text{dR}} &= \langle \nabla_{kl}F^*((\nabla_{ii}\omega_i)^{(p)}), \omega_j \rangle_{\text{dR}} + \langle F^*((\nabla_{ii}\omega_i)^{(p)}), \nabla_{kl}\omega_j \rangle_{\text{dR}} \\ &= \langle F^*((\nabla_{ii}\omega_i)^{(p)}), \nabla_{kl}\omega_j \rangle_{\text{dR}}. \end{aligned}$$

It follows from this that we have  $3 \times 4 = 12$  relations in the first claim.

The second claim follows from the first one with the commutativity of the differentials. For instance,

$$\nabla_{11}(b_{11}) = -\nabla_{11}\nabla_{12}(a_{21}) = -\nabla_{12}\nabla_{11}(a_{21}) = 0$$

and

$$\nabla_{12}(b_{12}) = -\nabla_{12}\nabla_{11}(a_{12}) = \nabla_{11}(b_{22}) = -\nabla_{11}\nabla_{22}(a_{22}) = -\nabla_{22}\nabla_{11}(a_{22}) = 0.$$

The remaining cases are left to readers.  $\square$

**3.3.1. Scalar valued case.** Fix a local basis  $\omega_1, \omega_2$  of  $\mathcal{E}$ . Recall the Hodge filtration  $\mathcal{E} = f_*\Omega_{\mathcal{A}/S_{N,p}}^1 \subset \mathbb{H}_{\text{dR}}^1(\mathcal{A}/S_{N,p})$ . Since we work locally on  $S_{N,p}^h$ , by Theorem 3.1, the Frobenius map splits the Hodge filtration. Hence we have obtained a canonical decomposition  $\mathbb{H}_{\text{dR}}^1(\mathcal{A}/S_{N,p}) = \mathcal{E} \oplus U$ ,  $U = \text{Im}(Fr^*)$ . We denote by  $R(U) = R_{\underline{k}}(U)$  for  $\underline{k} = (k_1, k_2)$  a canonical direct factor so that

$$\text{Sym}^{k_2}\left(\bigwedge^2 \mathbb{H}_{\text{dR}}^1(\mathcal{A}/S_{N,p})\right) \otimes_{\mathcal{O}_{S_{N,p}}} \text{Sym}^{k_1-k_2} \mathbb{H}_{\text{dR}}^1(\mathcal{A}/S_{N,p}) = \omega_{\underline{k}} \oplus R(U)$$

which is induced from the above decomposition of  $\mathbb{H}_{\text{dR}}^1(\mathcal{A}/S_{N,p})$ . For simplicity put  $\mathbb{H}_{\text{dR}}^1 := \mathbb{H}_{\text{dR}}^1(\mathcal{A}/S_{N,p})$ . Then we have an operator  $\tilde{\theta}$  which is given by

$$(3.23) \quad \begin{array}{ccc} \omega^{\otimes k} & \xrightarrow{\hookrightarrow} & \text{Sym}^k(\bigwedge^2 \mathbb{H}_{\text{dR}}^1) \xrightarrow{\nabla} \text{Sym}^k(\bigwedge^2 \mathbb{H}_{\text{dR}}^1) \otimes_{\mathcal{O}_{S_{N,p}}} \Omega_{S_{N,p}}^1 \\ & \searrow \tilde{\theta} & \downarrow KS^{-1} \\ & & \text{Sym}^k(\bigwedge^2 \mathbb{H}_{\text{dR}}^1) \otimes_{\mathcal{O}_{S_{N,p}}} \text{Sym}^2 \mathcal{E} \\ & & \downarrow \text{mod } R(U) \\ & & \omega^{\otimes k} \otimes_{\mathcal{O}_{S_{N,p}}} \text{Sym}^2 \mathcal{E}. \end{array}$$

Similarly we define operators  $\tilde{\theta}_i, i = 1, 2, 3$ :

$$(3.24) \quad \begin{array}{ccc} \omega^{\otimes k} \otimes_{\mathcal{O}_{S_{N,p}}} \text{Sym}^2 \mathcal{E} & \xrightarrow{\hookrightarrow} & \text{Sym}^k(\wedge^2 \mathbb{H}_{\text{dR}}^1) \otimes \text{Sym}^2(\mathbb{H}_{\text{dR}}^1) \\ \tilde{\theta}_1 \swarrow & & \downarrow KS^{-1} \circ \nabla \\ & \text{Sym}^k(\wedge^2 \mathbb{H}_{\text{dR}}^1) \otimes \text{Sym}^2(\mathbb{H}_{\text{dR}}^1) \otimes_{\mathcal{O}_{S_{N,p}}} \text{Sym}^2 \mathcal{E} & \\ & \downarrow \text{mod } R(U) & \\ & \omega^{\otimes k} \otimes_{\mathcal{O}_{S_{N,p}}} \text{Sym}^2 \mathcal{E} \otimes_{\mathcal{O}_{S_{N,p}}} \text{Sym}^2 \mathcal{E} & \\ \tilde{\theta}_2 \downarrow & & \downarrow p_3 \\ \omega^{\otimes k+2} & \xleftarrow{p_1} \omega^{\otimes k+1} \otimes_{\mathcal{O}_{S_{N,p}}} \text{Sym}^2 \mathcal{E} \xleftarrow{p_2} \omega^{\otimes k} \otimes_{\mathcal{O}_{S_{N,p}}} \text{Sym}^4 \mathcal{E} & \end{array}$$

where the three arrows  $p_1, p_2, p_3$  are given by an explicit, but non-canonical direct decomposition

$$(3.25) \quad \text{Sym}^2 \mathcal{E} \otimes_{\mathcal{O}_{S_{N,p}}} \text{Sym}^2 \mathcal{E} = \omega^{\otimes 2} \oplus (\omega \otimes_{\mathcal{O}_{S_{N,p}}} \text{Sym}^2 \mathcal{E}) \oplus \text{Sym}^4 \mathcal{E}$$

as  $\text{Aut}_{\mathcal{O}_{S_{N,p}}}(\mathcal{E}) \simeq GL_2(\mathcal{O}_{S_{N,p}})$ -modules. Let us explain the above decomposition. Fix a local basis  $e_1, e_2$  of  $\mathcal{E}$ . Put  $u_i = e_1^i e_2^{2-i}, i = 0, 1, 2$  which make up a local basis of  $\text{Sym}^2 \mathcal{E}$ . To avoid the confusion, we prepare another symbols  $v_i = e_1^i e_2^{2-i}, i = 0, 1, 2$  which play the same role. Then  $\{u_i \otimes v_j\}_{0 \leq i, j \leq 2}$  gives a local basis of  $\text{Sym}^2 \mathcal{E} \otimes_{\mathcal{O}_{S_{N,p}}} \text{Sym}^2 \mathcal{E}$ . As we will see later in Appendix A, For any  $x = \sum_{0 \leq i, j \leq 2} a_{ij} u_i \otimes v_j$  we define

$$(3.26) \quad \begin{aligned} p_1(x) &= \left(\frac{1}{3}a_{20} - \frac{1}{6}a_{11} + \frac{1}{3}a_{02}\right)(e_1 \wedge e_2)^2 \\ p_2(x) &= \frac{1}{2}(a_{21} - a_{12})e_1^2(e_1 \wedge e_2) + (a_{20} - a_{02})e_1 e_2(e_1 \wedge e_2) + \frac{1}{2}(a_{10} - a_{01})e_2^2(e_1 \wedge e_2) \\ p_3(x) &= a_{22}e_1^4 + (a_{21} + a_{12})e_1^3 e_2 + (a_{20} + a_{11} + a_{02})e_1^2 e_2^2 + (a_{10} + a_{01})e_1 e_2^3 + a_{00}e_2^4 \end{aligned}$$

which give (3.25) in local basis  $\{u_i \otimes v_j\}_{0 \leq i, j \leq 2}$ . We have used the assumption  $p \geq 5$  to get this decomposition. Otherwise it is no longer decomposable as  $GL_2(\mathcal{O}_{S_{N,p}})$ -modules.

We now compute the images of theta operators. Recall that we have worked locally on the ordinary locus  $S_{N,p}^h$ . Therefore we may put  $C = BA^{-1} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$ . Then we have

$$(3.27) \quad c_{11} = \frac{b_{11}a_{22} - b_{12}a_{21}}{\det(A)}, \quad c_{12} = \frac{-b_{11}a_{12} + b_{12}a_{11}}{\det(A)}, \quad c_{21} = \frac{b_{21}a_{22} - b_{22}a_{21}}{\det(A)}, \quad c_{22} = \frac{-b_{21}a_{12} + b_{22}a_{11}}{\det(A)}.$$

By (3.22) and Lemma 3.2-(2), we have

$$(3.28) \quad \begin{aligned} \nabla_{12}\omega_2 &= \nabla_{11}\omega_1 = \phi_1 - (c_{11}\omega_1 + c_{12}\omega_2), \\ \nabla_{12}\omega_1 &= \nabla_{22}\omega_2 = \phi_2 - (c_{21}\omega_1 + c_{22}\omega_2). \end{aligned}$$

For simplicity, put  $\omega_1^2 = \omega_1^{\otimes 2}$ ,  $\omega_1\omega_2 = \frac{1}{2}(\omega_1 \otimes \omega_2 + \omega_2 \otimes \omega_1)$ ,  $\omega_2^2 = \omega_2^{\otimes 2}$  (see (3.15)). Put  $d_{ij} := \langle \omega_i, \nabla \omega_j \rangle_{\text{dR}}$  for  $1 \leq i \leq j \leq 2$ . Then we have

**Proposition 3.5.** *Keep the notations above. Let  $F(\omega_1 \wedge \omega_2)^k$  be a local section of  $\omega^k$ . Then*

$$\tilde{\theta}(F(\omega_1 \wedge \omega_2)^k) = \begin{pmatrix} \nabla_{11}F - kc_{11}F \\ \nabla_{12}F - k(c_{12} + c_{21})F \\ \nabla_{22}F - kc_{22}F \end{pmatrix}^t \begin{pmatrix} (\omega_1 \wedge \omega_2)^k \omega_1^2 \\ (\omega_1 \wedge \omega_2)^k \omega_1 \omega_2 \\ (\omega_1 \wedge \omega_2)^k \omega_2^2 \end{pmatrix}.$$

*Proof.* By definition,

$$\begin{aligned} KS^{-1} \circ \nabla(F(\omega_1 \wedge \omega_2)^k) &= \sum_{1 \leq i \leq j \leq 2} (\nabla_{ij} F(\omega_1 \wedge \omega_2)^k) KS^{-1}(d_{ij}) \\ &= \sum_{1 \leq i \leq j \leq 2} (\nabla_{ij} F(\omega_1 \wedge \omega_2)^k) \omega_i \omega_j. \end{aligned}$$

Therefore we have only to compute  $\nabla_{ij} F(\omega_1 \wedge \omega_2)^k$  modulo  $R(U)$ . By direct computation with (3.28),

$$\begin{aligned} \nabla_{11}(\omega_1 \wedge \omega_2) &= \nabla_{11}\omega_1 \wedge \omega_2 + \omega_1 \wedge \nabla_{11}\omega_2 \\ &= \phi_1 \wedge \omega_2 - c_{11}\omega_1 \wedge \omega_2 \equiv -c_{11}\omega_1 \wedge \omega_2 \pmod{R(U)}. \end{aligned}$$

It follows from this that

$$\begin{aligned} \nabla_{11}(F(\omega_1 \wedge \omega_2)^k) &= \nabla_{11}(F)(\omega_1 \wedge \omega_2)^k + kF(\omega_1 \wedge \omega_2)^{k-1} \nabla_{11}(\omega_1 \wedge \omega_2) \\ &\equiv (\nabla_{11}F - kc_{11}F)(\omega_1 \wedge \omega_2)^k \pmod{R(U)}. \end{aligned}$$

We will have the same results for remaining cases by using

$$\nabla_{12}(\omega_1 \wedge \omega_2) \equiv -(c_{12} + c_{21})\omega_1 \wedge \omega_2 \pmod{R(U)}, \quad \nabla_{22}(\omega_1 \wedge \omega_2) \equiv -c_{22}\omega_1 \wedge \omega_2 \pmod{R(U)}.$$

□

The following lemma will be used later. We omit the proof.

**Lemma 3.6.** *Keep the notations above. Then under modulo  $R(U)$ ,*

$$\begin{aligned} \nabla_{11}(\omega_1^2) &\equiv -2c_{11}\omega_1^2 - 2c_{12}\omega_1\omega_2, \quad \nabla_{12}(\omega_1^2) \equiv -2c_{21}\omega_1^2 - 2c_{22}\omega_1\omega_2, \quad \nabla_{22}(\omega_1^2) = 0, \\ \nabla_{11}(\omega_1\omega_2) &\equiv -c_{11}\omega_1\omega_2 - c_{12}\omega_2^2, \quad \nabla_{12}(\omega_1\omega_2) \equiv -c_{11}\omega_1^2 - (c_{12} + c_{21})\omega_1\omega_2 - c_{22}\omega_2^2, \\ \nabla_{22}(\omega_1\omega_2) &\equiv -c_{21}\omega_1^2 - c_{22}\omega_1\omega_2, \\ \nabla_{11}(\omega_2^2) &\equiv 0, \quad \nabla_{12}(\omega_2^2) \equiv -2c_{11}\omega_1\omega_2 - 2c_{12}\omega_2^2, \quad \nabla_{22}(\omega_2^2) \equiv -2c_{21}\omega_1\omega_2 - 2c_{22}\omega_2^2. \end{aligned}$$

Next we compute the image of each  $\tilde{\theta}_i, i = 1, 2, 3$ . Let  $F_{11}(\omega_1 \wedge \omega_2)^k \omega_1^2, F_{12}(\omega_1 \wedge \omega_2)^k \omega_1 \omega_2, F_{22}(\omega_1 \wedge \omega_2)^k \omega_2^2$  be local sections of  $\omega^k \otimes_{\mathcal{O}_{S_{N,p}}} \text{Sym}^2 \mathcal{E}$ . Put

$$h = F_{11}(\omega_1 \wedge \omega_2)^k \omega_1^2 + F_{12}(\omega_1 \wedge \omega_2)^k \omega_1 \omega_2 + F_{22}(\omega_1 \wedge \omega_2)^k \omega_2^2.$$

Let  $u_i, v_j, 0 \leq i, j \leq 2$  be the basis to define (3.26) with respect to  $e_1 = \omega_1, e_2 = \omega_2$ . We denote by  $\sum_{0 \leq i, j \leq 2} a_{ij}(\omega_1 \wedge \omega_2)^k u_i \otimes v_j, a_{ij} \in \mathcal{O}_{S_{N,p}^h}$  the image of  $KS^{-1} \circ \nabla(h)$  under the projection to  $\omega^{\otimes k} \otimes_{\mathcal{O}_{S_{N,p}}} \text{Sym}^2 \mathcal{E} \otimes_{\mathcal{O}_{S_{N,p}}} \text{Sym}^2 \mathcal{E}$ .

**Proposition 3.7.** *Keep the notations above. Each  $a_{ij}$  is given as follows:*

$$\begin{aligned} a_{22} &= \nabla_{11}(F_{11}) - (k+2)c_{11}F_{11}, & a_{21} &= \nabla_{12}(F_{11}) - (k(c_{12} + c_{21}) + 2c_{21})F_{11} - c_{11}F_{12}, \\ a_{12} &= \nabla_{11}(F_{12}) - (k+1)c_{11}F_{12} - 2c_{12}F_{11}, \\ a_{20} &= \nabla_{22}(F_{11}) - kc_{22}F_{11} - c_{21}F_{12}, & a_{11} &= \nabla_{12}(F_{12}) - (k+1)(c_{12} + c_{21})F_{12} - 2c_{22}F_{11} - 2c_{11}F_{22}, \\ a_{02} &= \nabla_{11}(F_{22}) - kc_{11}F_{22} - c_{12}F_{12}, \\ a_{10} &= \nabla_{22}(F_{12}) - (k+1)c_{22}F_{12} - 2c_{21}F_{22}, & a_{01} &= \nabla_{12}(F_{22}) - (k(c_{12} + c_{21}) + 2c_{12})F_{22} - c_{22}F_{12}, \\ a_{00} &= \nabla_{22}(F_{22}) - (k+2)c_{22}F_{22}. \end{aligned}$$

*Proof.* By definition, for  $1 \leq i \leq j \leq 2$ ,

$$\begin{aligned} KS^{-1} \circ \nabla(F_{ij}(\omega_1 \wedge \omega_2)^k \omega_i \omega_j) &= \sum_{1 \leq k \leq l \leq 2} (\nabla_{kl} F(\omega_1 \wedge \omega_2)^k \omega_i \omega_j) KS^{-1}(d_{kl}) \\ &= \sum_{1 \leq i \leq j \leq 2} (\nabla_{kl} F_{ij}(\omega_1 \wedge \omega_2)^k \omega_i \omega_j) \otimes \omega_k \omega_l. \end{aligned}$$

Therefore we have only to compute

$$\nabla_{kl} F_{ij}(\omega_1 \wedge \omega_2)^k \omega_i \omega_j = (\nabla_{kl} F_{ij})(\omega_1 \wedge \omega_2)^k \omega_i \omega_j + F_{ij}(\nabla_{kl}(\omega_1 \wedge \omega_2)^k) \omega_i \omega_j + (\omega_1 \wedge \omega_2)^k \nabla_{kl}(\omega_i \omega_j).$$

By using Lemma 3.6 for the third term (see the proof of Proposition 3.5 for the second term), we easily see the claim.  $\square$

**Proposition 3.8.** *Keep the notation above. Put  $\tilde{\Theta} := \tilde{\theta}_1 \circ \tilde{\theta} : \omega^k \rightarrow \omega^{k+2}$  which is defined on  $S_{N,p}^h$ . Let  $F(\omega_1 \wedge \omega_2)^k$  be a local section of  $\omega^k$ . Assume that  $\nabla_{ij}$ 's are commutative each other. Then*

$$\begin{aligned} (\det A) \cdot \tilde{\Theta}(F) &= \frac{2}{3} \det A \cdot \det \begin{pmatrix} \nabla_{11} & \frac{1}{2} \nabla_{12} \\ \frac{1}{2} \nabla_{12} & \nabla_{22} \end{pmatrix} (F) + \frac{2k(2k-1)}{9} F \cdot \det \begin{pmatrix} \nabla_{11} & \frac{1}{2} \nabla_{12} \\ \frac{1}{2} \nabla_{12} & \nabla_{22} \end{pmatrix} (\det A) \\ &\quad + \frac{2k-1}{3} \left( \nabla_{11}(F) \nabla_{22}(\det A) - \frac{1}{2} \nabla_{12}(F) \nabla_{12}(\det A) + \nabla_{22}(F) \nabla_{11}(\det A) \right) \end{aligned}$$

$$\text{where } \det \begin{pmatrix} \nabla_{11} & \frac{1}{2} \nabla_{12} \\ \frac{1}{2} \nabla_{12} & \nabla_{22} \end{pmatrix} (F) := (\nabla_{11} \nabla_{22} - \frac{1}{4} \nabla_{12}^2) F.$$

*Proof.* By the assumption of the commutativity of  $\nabla_{ij}$ , we see that

$$\langle \nabla_{11} \omega_1, \nabla_{22} \omega_2 \rangle_{dR} = 0.$$

It follows from this that  $c_{12} = c_{21}$ . By Proposition 3.5 and 3.7, as a first step we have

$$\begin{aligned} \tilde{\Theta}(F) &= \frac{2}{3} \det \begin{pmatrix} \nabla_{11} & \frac{1}{2} \nabla_{12} \\ \frac{1}{2} \nabla_{12} & \nabla_{22} \end{pmatrix} (F) + F \left\{ -\frac{k}{3} (\nabla_{22} c_{11} - \nabla_{12} c_{12} + \nabla_{11} c_{22}) + \frac{2k(k-1)}{3} (c_{11} c_{22} - c_{12}^2) \right\} \\ &\quad + \frac{2k-1}{3} (c_{12} \nabla_{12}(F) - c_{11} \nabla_{22}(F) - c_{22} \nabla_{11}(F)). \end{aligned}$$

By the assumption of the commutativity of  $\nabla_{ij}$ 's and Proposition 3.4, one has

$$(3.29) \quad \nabla_{22} c_{11} - \nabla_{12} c_{12} + \nabla_{11} c_{22} = -\frac{\det B}{\det A}.$$

It follows from this that the second term becomes  $\frac{k(2k-1)}{3} \cdot \frac{\det B}{\det A}$ . On the other hand, by Proposition 3.4 again,  $\det \begin{pmatrix} \nabla_{11} & \frac{1}{2} \nabla_{12} \\ \frac{1}{2} \nabla_{12} & \nabla_{22} \end{pmatrix} (\det A) = \frac{3}{2} \det B$ . Putting these together, the second term becomes one as in the claim. The third term is computed by using

$$(3.30) \quad \nabla_{11}(\det A) = -c_{11} \det A, \quad \nabla_{12}(\det A) = -2c_{12} \det A, \quad \nabla_{22}(\det A) = -c_{22} \det A.$$

This gives us the claim.  $\square$

Put  $\Theta := H_{p-1} \cdot \tilde{\Theta}$ ,  $\theta := H_{p-1} \cdot \tilde{\theta}$  and  $\theta_i := H_{p-1} \cdot \tilde{\theta}_i$ ,  $i = 1, 2, 3$ . We will check the holomorphy is preserved under these operators. However to apply Proposition 3.7 and Proposition 3.8, we have to check the commutativity of the dual basis  $D_{ij}$ ,  $1 \leq i \leq j \leq 2$ . So far we do not know how to do that because we work on the field of positive characteristic. To overcome this situation we move to the localization at each point apply to the local deformation of each point which is called the generic square zero deformation by Katz (see p. 180-185 of [42]).



Let  $S = \overline{\mathbb{F}}_p[[t_{11}, t_{12}, t_{22}]]$  be the formal power series over  $\overline{\mathbb{F}}_p$  with the three variables and  $m_S$  be the maximal ideal of  $S$ . Put  $R = S/m_S^2$ . Fix a point  $A \in S_{N,p}(\overline{\mathbb{F}}_p)$  and a basis  $\omega'_1, \omega'_2$  of  $H^0(A, \Omega_A^1)$ . Let  $\eta'_i \in H^1(A, \mathcal{O}_A)$  be the dual element of  $\omega'_i$ . Let  $s_{ij} \in \text{Hom}_{\overline{\mathbb{F}}_p}(H^0(A, \Omega_A^1), H^1(A, \mathcal{O}_A))$  be the element sending  $\omega'_i$  to  $\eta'_j$  and  $\omega'_k$  to 0 if  $k \neq i$ . Let  $\mathcal{X}$  be the deformation of  $A$  to  $R$  corresponding to

$$s_{11}t_{11} + s_{12}t_{12} + s_{22}t_{22} \in \text{Hom}_{\overline{\mathbb{F}}_p}(H^0(A, \Omega_A^1), H^1(A, \mathcal{O}_A)) \otimes_{\overline{\mathbb{F}}_p} m_R$$

which we call it the generic zero square deformation (see SUBLEMMA 8 at p.180 of [42]). We denote by  $f_{\mathcal{X}} : \mathcal{X} \rightarrow \text{Spec} R$  the corresponding abelian scheme. Put  $D_{ij} = \frac{\partial}{\partial t_{ij}} \in \text{Der}(R)$ ,  $1 \leq i, j \leq 2$ . Let  $\omega_{i,\mathcal{X}}$  be any lift of  $\omega_i$  to  $f_{\mathcal{X}*}\Omega_{\mathcal{X}/\text{Spec} R}^1$ . Then by the formation of the deformation space, we see that

$$\langle \omega_{i,\mathcal{X}}, \nabla(D_{kl})\omega_{j,\mathcal{X}} \rangle_{\text{dR}} = \delta_{ik}\delta_{lj}, \quad 1 \leq i \leq j \leq 2, \quad 1 \leq i \leq j \leq 2.$$

Clearly  $\nabla(D_{kl})$ 's are commutative each other. Hence universality of  $\mathcal{A}/S_{N,p}$ , to study local behavior of the functions on  $S_{N,p}$ , we may work on  $\mathcal{X}/\text{Spec} R$  and we can apply the results in Section 3.1 to  $\mathcal{X}/\text{Spec} R$ . This means that if we define the functions  $a_{ij,\mathcal{X}}, b_{ij,\mathcal{X}} \in R$  as (3.21) for  $\mathcal{X}/\text{Spec} R$ , these are nothing but the images of  $a_{ij}, b_{ij} \in \mathcal{O}_{S_{N,p}}$  under the map  $\mathcal{O}_{S_{N,p}} \rightarrow R$  coming from the universality.

**Proposition 3.9.** *Keep the notations above. Recall the Hasse invariant  $H_{p-1} = \det A \in H^0(S_{N,p}, \omega^{p-1})$ . Put  $\Theta := H_{p-1} \cdot \tilde{\Theta}$ ,  $\theta := H_{p-1} \cdot \tilde{\theta}$  and  $\theta_i := H_{p-1} \cdot \tilde{\theta}_i$ ,  $i = 1, 2, 3$ . Then the following properties are satisfied:*

- (1) Let  $F \in M_k(\Gamma(N), \overline{\mathbb{F}}_p)$ . Assume  $\Theta(F)$  and  $\theta(F)$  are not identically zero, then
  - (a)  $\Theta(F) \in M_{k+p+1}(\Gamma(N), \overline{\mathbb{F}}_p)$  and  $T(\ell^i)\Theta(F) = \ell^{2i}T(\ell^i)F$ ,
  - (b)  $\theta(F) \in M_{(k+p+1, k+p-1)}(\Gamma(N), \overline{\mathbb{F}}_p)$  and  $\lambda_{\theta(F)}(\ell^i) = \ell^i \lambda_F(\ell^i)$  where  $\lambda_G(\ell^i)$  stands for Hecke eigenvalue of  $G$  for  $T(\ell^i)$  (see the last part of Section 2.5 with (2.8)),
- (2) Let  $F \in M_{(k+2, k)}(\Gamma(N), \overline{\mathbb{F}}_p)$  and  $i \in \{1, 2, 3\}$ . Assume that  $\theta_i(F)$  is not identically zero, then  $\theta_i(F) \in M_{(k+p+i, k+p+2-i)}(\Gamma(N), \overline{\mathbb{F}}_p)$  and  $\lambda_{\theta_i(F)}(\ell^i) = \ell^i \lambda_F(\ell^i)$ .

*Proof.* The weight correspondence in each case is obvious by definition. We postpone to give a proof for Hecke eigenvalues until the time being (see the vector valued case) because this should be proved under more general situation.

What we have to do is to check that the holomorphy is preserved under these operators. Except for  $\Theta$ , By Proposition 3.7 the holomorphy is obvious since  $\det(A)c_{ij}$ 's are holomorphic, hence elements in the total space  $S_{N,p}$ . For  $\Theta$ , by using generic square zero deformation and Proposition 3.8, we see that  $\Theta(F) = \det(A) \cdot \tilde{\Theta}(F)$  does not have any denominator. The point here is a priori we know  $\det(A)^2 \cdot \tilde{\Theta}(F)$  is holomorphic on  $S_{N,p}$  without using localization, but thanks to the symmetry  $c_{12} = c_{21}$  after the localization, we know  $\det(A) \cdot \tilde{\Theta}(F)$  is holomorphic. This gives us the claims.  $\square$

**Remark 3.10.** *The definition of  $\theta$  works even when  $p = 3$ .*

To study  $q$ -expansion of the image of mod  $p$  Siegel modular forms under the above operators we need to compute Hasse invariant of the Mumford's semi-abelian scheme. Recall the Mumford's semi-abelian scheme  $\pi : \mathcal{A}^{\text{Mum}} \rightarrow T_N$  (2.19) and consider its (relative) base change to  $\overline{\mathbb{F}}_p$  which is denoted by  $\pi_p : \mathcal{A}_p^{\text{Mum}} = \mathcal{A}^{\text{Mum}} \otimes \overline{\mathbb{F}}_p \rightarrow T_{N,p} := T_N \otimes \overline{\mathbb{F}}_p$ . Fix a canonical invariant forms

$\omega_{\text{can},1} = \frac{dt_1}{t_1}, \omega_{\text{can},2} = \frac{dt_2}{t_2}$  on  $\mathcal{A}_p^{\text{Mum}}$  which will be a basis of  $\pi_{p*}\Omega_{\mathcal{A}_p^{\text{Mum}}/T_{N,p}}^1$ . Recall the periods  $q_{ij}, 1 \leq i \leq j \leq 2$  of this abelian variety.

**Proposition 3.11.** *The following statements are satisfied:*

- (1) *for any  $1 \leq i \leq j \leq 2$ ,  $\langle \omega_{\text{can},i}, \nabla \omega_{\text{can},j} \rangle_{\text{dR}} = dq_{ij}$  where  $\nabla$  is the Gauss-Manin connection with respect to  $\pi_p$ .*
- (2) *The Hasse matrix of the Mumford's semi-abelian scheme  $\mathcal{A}^{\text{Mum}} \otimes \overline{\mathbb{F}}_p$  is the identity matrix with respect to the canonical basis  $\omega_{\text{can},1} = \frac{dt_1}{t_1}, \omega_{\text{can},2} = \frac{dt_2}{t_2}$ . In particular,  $q$ -expansion of the Hasse invariant  $H_{p-1}$  satisfies  $H_{p-1}(q) = 1$ .*

*Proof.* The first claim follows from the fact that the Gauss-Manin connection is of formation compatible with any base change. In fact, before tensoring  $\overline{\mathbb{F}}_p$ , we once moved to  $p$ -adic setting and then apply Theorem 1, p.434 of [19].

We now prove the second claim. Let  $\eta_{\text{can},i}$  be the element of  $R^1\pi_{p*}\mathcal{O}_{\mathcal{A}_p^{\text{Mum}}}$  corresponding to  $\omega_{\text{can},i}$ . Then it is well-known (cf. p.167 of [26]) that for any local sections  $x \in R^1\pi_{p*}\mathcal{O}_{\mathcal{A}_p^{\text{Mum}}}$  and  $y \in \pi_{p*}\Omega_{\mathcal{A}_p^{\text{Mum}}/T_{N,p}}^1$ , one has

$$\langle F^*(x^{(p)}), y \rangle_{\text{dual}} = \langle x, V^*(y)^{(p^{-1})} \rangle_{\text{dual}}^{(p)}$$

where  $V : (\mathcal{A}_p^{\text{Mum}})^{(p)} \rightarrow \mathcal{A}_p^{\text{Mum}}$  stands for the Verschiebung morphism and  $V^*(*)^{(p^{-1})}$  is defined by the composition of the pullback  $V^* : \pi_{p*}\Omega_{\mathcal{A}_p^{\text{Mum}}/T_{N,p}}^1 \rightarrow (\pi_p)^{(p)}_*\Omega_{(\mathcal{A}_p^{\text{Mum}})^{(p)}/T_{N,p}}^1$  and the dual of the absolute Frobenius  $F_{\text{abs}}^\vee : (\pi_p)^{(p)}_*\Omega_{(\mathcal{A}_p^{\text{Mum}})^{(p)}/T_{N,p}}^1 \rightarrow \pi_{p*}(\Omega_{\mathcal{A}_p^{\text{Mum}}/T_{N,p}}^1)^{(p)}$  with respect to the polarization  $\lambda$ . The superscript of  $\langle x, V^*(y)^{(p^{-1})} \rangle_{\text{dual}}^{(p)}$  means the image of  $\langle x, V^*(y)^{(p^{-1})} \rangle_{\text{dual}}$  under the natural map  $\mathcal{O}_{T_{N,p}}^{(p)} = \mathcal{O}_{T_{N,p}} \otimes_{\mathcal{O}_{T_{N,p}}, F_{\text{abs}}} \mathcal{O}_{T_{N,p}} \rightarrow \mathcal{O}_{T_{N,p}}$ . Since we have the identifications between tangent spaces  $\text{Tan}(\mathcal{A}_p^{\text{Mum}}) := \pi_{p*}\Omega_{\mathcal{A}_p^{\text{Mum}}/T_{N,p}}^1 = \text{Tan}(\mathcal{A}_p^{\text{Mum}}[p])$ , by using invariant forms  $\omega_1, \omega_2$ , it is naturally identified with  $\text{Tan}(\mu_p/T_{N,p})^{\oplus 2}$  where  $\mu_p/T_{N,p} = \text{Spec } \overline{\mathbb{F}}_p[x]/(x^p - 1) \times_{\text{Spec } \overline{\mathbb{F}}_p} T_{N,p}$ . Here we implicitly used a canonical base  $\frac{dx}{x}$  of  $\Omega_{\mu_p/T_{N,p}}^1$  to identify. Hence the dual  ${}^tV^*(*)^{(p^{-1})}$  is naturally identified with the map on  $\text{Tan}(\mu_p/T_{N,p})^{\oplus 2}$  which also comes from Verschiebung on  $(\mu_p/T_{N,p})^{\times 2}$ . However it is well-known that it induces the identity map. Hence  $V^*(*)^{(p^{-1})} = 1$  with respect to  $\omega_{\text{can},1}, \omega_{\text{can},2}$ . Plugging this into above formula, one has

$$\langle F^*(\eta_{\text{can},i}^{(p)}), \omega_{\text{can},j} \rangle_{\text{dual}} = \langle \eta_{\text{can},i}, V^*(\omega_{\text{can},j})^{(p^{-1})} \rangle_{\text{dual}}^{(p)} = \langle \eta_{\text{can},i}, \omega_{\text{can},j} \rangle_{\text{dual}}^{(p)} = \delta_{ij}$$

where  $\delta_{ij}$  is the Kronecker's delta function. Then by non-degeneracy of the pairing, one has  $F^*(\eta_{\text{can},i}^{(p)}) = \eta_{\text{can},i}$  which gives us the claim.  $\square$

**3.3.2. Vector valued case.** Let us first assume that  $\underline{k} = (k_1, k_2), k_1 \geq k_2 \geq 1$  satisfies  $p > k_1 - k_2 + 2$ . Under this assumption we have a non-canonical decomposition

$$(3.31) \quad \text{Sym}^{k_1-k_2}\mathcal{E} \otimes_{\mathcal{O}_{S_{N,p}}} \text{Sym}^2\mathcal{E} = (\text{Sym}^{k_1-k_2-2}\mathcal{E} \otimes_{\mathcal{O}_{S_{N,p}}} \omega^{\otimes 2}) \oplus (\text{Sym}^{k_1-k_2}\mathcal{E} \otimes_{\mathcal{O}_{S_{N,p}}} \omega) \oplus \text{Sym}^{k_1-k_2+2}\mathcal{E}$$

as  $GL_2(\mathcal{O}_{S_{N,p}})$ -modules (see Appendix A). When  $k_1 - k_2 = 1$ , the decomposition

$$(3.32) \quad \mathcal{E} \otimes_{\mathcal{O}_{S_{N,p}}} \text{Sym}^2\mathcal{E} = (\text{Sym}^3\mathcal{E} \otimes_{\mathcal{O}_{S_{N,p}}} \omega) \oplus \mathcal{E}$$

is given by sending  $\sum_{\substack{1 \leq i \leq 2 \\ 0 \leq j \leq 2}} a_{ij} w_i \otimes v_j$ ,  $v_j = \omega_1^j \omega_2^{2-j}$  to

$$(3.33) \quad \{a_{12}\omega_1^3 + (a_{11} + a_{02})\omega_1^2\omega_2 + (a_{10} + a_{01})\omega_1\omega_2^2 + a_{02}\omega_2^3\}(\omega_1 \wedge \omega_2) \oplus \left(\frac{a_{11}}{3} - \frac{2a_{02}}{3}\right)\omega_1 + \left(\frac{2a_{10}}{3} - \frac{a_{01}}{3}\right)\omega_2.$$

As (3.24), one has

$$(3.34) \quad \begin{array}{ccccc} & \omega_{\underline{k}} = \omega^{\otimes k_2} \otimes_{\mathcal{O}_{S_{N,p}}} \text{Sym}^{k_1-k_2} \mathcal{E} & & & \\ & \nwarrow \tilde{\theta}_1^k & \downarrow \tilde{\theta}_2^k & \searrow \tilde{\theta}_3^k & \\ \omega_{(k_1, k_2+2)} & & \omega_{(k_1+1, k_2+1)} & & \omega_{(k_1+2, k_2)} \end{array}$$

We should remember that these operators are depending on (3.31). Then we have the following:

**Proposition 3.12.** *Keep the notations above. Recall the Hasse invariant  $H_{p-1} = \det A \in H^0(S_{N,p}, \omega^{p-1})$ .*

*Put  $\theta_i = \theta_i^k := H_{p-1} \cdot \tilde{\theta}_i^k$ ,  $i = 1, 2, 3$ . If  $F \in M_{\underline{k}}(\Gamma(N), \overline{\mathbb{F}}_p)$ , then*

- (1)  $\theta_1^k(F) \in M_{(k_1+p-1, k_2+p+1)}(\Gamma(N), \overline{\mathbb{F}}_p)$ ,  $\theta_2^k(F) \in M_{(k_1+p, k_2+p)}(\Gamma(N), \overline{\mathbb{F}}_p)$ , and  $\theta_3^k(F) \in M_{(k_1+p+1, k_2+p-1)}(\Gamma(N), \overline{\mathbb{F}}_p)$ ,
- (2) *if  $F$  is a Hecke eigenform, then  $\lambda_{\theta_j^k(F)}(\ell^i) = \ell^i \lambda_F(\ell^i)$ ,  $j = 1, 2, 3$  provided if it is not identically zero.*

*Proof.* We first show that for any  $F \in M_{\underline{k}}(\Gamma(N), \overline{\mathbb{F}}_p)$ ,  $H_{p-1} \cdot KS^{-1} \circ \nabla(F) \bmod R(U)$  is a Siegel modular form  $G$  of weight  $\omega_{(k_1+p-1, k_2+p-1)} \otimes_{S_{N,p}} \text{Sym}^2 \mathcal{E}$ , namely it belongs to  $H^0(S_{N,p}, \omega_{(k_1+p-1, k_2+p-1)} \otimes_{S_{N,p}} \text{Sym}^2 \mathcal{E})$ . To see this we compute  $G$  explicitly. Choose a local basis  $\omega_1, \omega_2$  of  $\mathcal{E}$ . Then  $\delta_n := \omega_1^{(k_1-k_2)-n} \omega_2^n (\omega_1 \wedge \omega_2)^{k_2}$ ,  $0 \leq n \leq k_1 - k_2$  make up a local basis of  $\omega_{\underline{k}}$ . Using this, we have a local expression  $F = \sum_{i=0}^{k_1-k_2} F_i \delta_i$ . We proceed as Proposition 3.5:

$$\begin{aligned} KS^{-1} \circ \nabla(F) &= \sum_{n=0}^{k_1-k_2} \sum_{1 \leq i \leq j \leq 2} \nabla_{ij}(F_n \delta_n) \otimes KS^{-1}(d_{ij}), \quad d_{ij} = \langle \omega_i, \nabla \omega_j \rangle_{\text{dR}} \\ &= \sum_{n=0}^{k_1-k_2} \sum_{1 \leq i \leq j \leq 2} \nabla_{ij}(F_n \delta_n) \otimes \omega_i \omega_j \\ &= \sum_{n=0}^{k_1-k_2} \sum_{1 \leq i \leq j \leq 2} (\nabla_{ij}(F_n) \delta_n + F_n \nabla_{ij}(\delta_n)) \otimes \omega_i \omega_j. \end{aligned}$$

Therefore we have only to compute  $\nabla_{ij}(F \delta_n)$  modulo  $R(U)$ . By direct computation, one has

$$(3.35) \quad \begin{aligned} \nabla_{11}(\delta_n) &\equiv -(k_1 - n)c_{11}\delta_n - (k_1 - k_2 - n)c_{12}\delta_{n+1}, \\ \nabla_{12}(\delta_n) &\equiv -nc_{11}\delta_{n-1} - ((k_1 - n)c_{21} + (k_2 + n)c_{12})\delta_n - (k_1 - k_2 - n)c_{22}\delta_{n+1}, \\ \nabla_{22}(\delta_n) &\equiv -nc_{21}\delta_{n-1} - (k_2 + n)c_{22}\delta_n. \end{aligned}$$

It follows from this with (3.27) that  $G$  is holomorphic since so is  $\det(A)c_{ij}$ ,  $1 \leq i, j \leq 2$  (see also (3.27)).

**Remark 3.13.** *Let  $R$  be a domain with characteristic  $p > 0$ . Let  $\rho_k = \text{Sym}^k St_2(R)$  and we denote by  $V(k)$  its representation space. We also denote by  $V(k, \ell)$  the representation space of  $\rho_k \otimes \det^\ell St_2$  and  $V(k)^{p^s}$  the  $s$ -th Frobenius twist of  $V(k)$ , that is the action is given by  $\rho_k \left( \begin{pmatrix} a^{p^s} & b^{p^s} \\ c^{p^s} & d^{p^s} \end{pmatrix} \right)$  for*

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(R)$ . When  $r := k_1 - k_2 = p - 1$  or  $r = p - 2$ , by direct calculation, we have the following exact sequence

$$0 \longrightarrow V(p-3) \longrightarrow V(p-1) \otimes V(2) \longrightarrow V(1)^{(p)} \otimes V(1) \oplus V(p-1, 1) \oplus V(p-3, 2) \longrightarrow 0$$

and

$$0 \longrightarrow V(p-2) \longrightarrow V(p-2) \otimes V(2) \longrightarrow V(1)^{(p)} \oplus V(p-2, 1) \oplus V(p-4, 2) \longrightarrow 0.$$

The projection to  $V(r-2, 2)$  splits and it is given by the same basis as in case  $r < p-2$ . So the definition of  $\theta_1^{(k_1, k_2)}$  for  $k_1 - k_2 = p-1$  or  $k_1 - k_2 = p-2$  still makes sense. This fact will be used later.

Next we compute the Hecke action on  $G$ . Assume  $F$  is a Hecke eigenform with  $q$ -expansion  $F(q) = \sum_{T \in \mathcal{S}(\mathbb{Z})_{\geq 0}} A_F(T) q_N^T = \sum_{n=0}^{k_1-k_2} \sum_{T \in \mathcal{S}(\mathbb{Z})_{\geq 0}} A_{F_n}(T) q_N^T \delta_n$  (Here we discuss about only one component of RHS of (2.20)). Note that  $A_F(T) = \sum_{n=0}^{k_1-k_2} A_{F_n}(T) \delta_n \in \text{Sym}^{k_1-k_2} St_2(\overline{\mathbb{F}}_p)$ . Since Hasse matrix for the Mumford semi-abelian scheme is identity with respect to the canonical basis related to the periods  $q_{kl}$ ,  $b_{11}(q) = 0$ , hence all  $c_{ij}(q)$ ,  $1 \leq i \leq j \leq 2$  are zero. This means that

$$G(q) = \sum_{T \in \mathcal{S}(\mathbb{Z})_{\geq 0}} A_G(T) q_N^T, \quad A_G(T) := \sum_{n=0}^{k_1-k_2} \frac{1}{N} A_{F_n}(T) \delta_n \otimes T$$

where we identify  $T = \begin{pmatrix} a_{11} & \frac{1}{2}a_{12} \\ \frac{1}{2}a_{12} & a_{22} \end{pmatrix}$  with the vector  $a_{11}e_1^2 + a_{12}e_1e_2 + a_{22}e_2^2$  in  $\text{Sym}^2 St_2(\mathbb{Z})$ .

Applying the formula (2.12) with  $(k'_1, k'_2) = (2 + (p-1), p-1)$ , we have

$$\begin{aligned} \lambda_F(\ell^i) A_G(T) &= A_G(\ell^i; T) \\ &= \sum_{\substack{\alpha+\beta+\gamma=i \\ \alpha, \beta, \gamma \geq 0}} \chi_1(\ell^\beta) \chi_2(\ell^\gamma) \ell^{\beta((k_1+2+p-1)-2)+\gamma((k_1+2+p-1)+k_2+p-1-3)} \times \\ &\quad \sum_{\substack{U \in R(\ell^\beta) \\ a_U \equiv 0 \pmod{\ell^{\beta+\gamma}} \\ b_U \equiv c_U \equiv 0 \pmod{\ell^\gamma}}} (\rho_{k_1-k_2} \otimes \rho_2) \left( \left( \begin{pmatrix} 1 & 0 \\ 0 & \ell^\beta \end{pmatrix} U \right)^{-1} \right) \left\{ A_F \left( \ell^\alpha \begin{pmatrix} a_U \ell^{-\beta-\gamma} & \frac{b_U \ell^{-\gamma}}{2} \\ \frac{b_U \ell^{-\gamma}}{2} & c_U \ell^{\beta-\gamma} \end{pmatrix} \right) \right. \\ &\quad \left. \otimes \rho_2 \left( \ell^\alpha \begin{pmatrix} a_U \ell^{-\beta-\gamma} & \frac{b_U \ell^{-\gamma}}{2} \\ \frac{b_U \ell^{-\gamma}}{2} & c_U \ell^{\beta-\gamma} \end{pmatrix} \right) T \right\} \\ &= \ell^{\alpha+\beta+\gamma} \sum_{\substack{\alpha+\beta+\gamma=i \\ \alpha, \beta, \gamma \geq 0}} \chi_1(\ell^\beta) \chi_2(\ell^\gamma) \ell^{\beta(k_1-2)+\gamma(k_1+k_2-3)} \times \\ &\quad \sum_{\substack{U \in R(\ell^\beta) \\ a_U \equiv 0 \pmod{\ell^{\beta+\gamma}} \\ b_U \equiv c_U \equiv 0 \pmod{\ell^\gamma}}} \rho_{k_1-k_2} \left( \left( \begin{pmatrix} 1 & 0 \\ 0 & \ell^\beta \end{pmatrix} U \right)^{-1} \right) A_F \left( \ell^\alpha \begin{pmatrix} a_U \ell^{-\beta-\gamma} & \frac{b_U \ell^{-\gamma}}{2} \\ \frac{b_U \ell^{-\gamma}}{2} & c_U \ell^{\beta-\gamma} \end{pmatrix} \right) \otimes T \\ &= \ell^i \lambda_F(\ell^i) A_G(T). \end{aligned}$$

□

We now turn to give a proof of Proposition 3.9.

*Proof.* We have only to show the claim (a)-(1). Other cases have been done already by Proposition 3.12. By the proof of Proposition 3.12, we have

$$\Theta(F)(q) = \frac{2}{3} \sum_{T \in \mathcal{S}(\mathbb{Z})_{\geq 0}} \frac{1}{N^2} \det(T) A_F(T) q_N^T.$$

Then the claim follows easily by the formula (2.20).  $\square$

**Remark 3.14.** In [7], Böcherer and Nagaoka studied “Ramanujan differential” for a mod  $p$  Siegel modular form of general degree  $g$  which comes from a classical form. When  $g = 2$ , we extend their result for which the forms are not necessarily coming from classical forms.

On the other hand, for any mod  $p$  Siegel modular form  $F$  of a parallel weight  $k$  of degree 2 with  $F(q) = \sum_{T \in \mathcal{S}(\mathbb{Z})_{\geq 0}} A_F(T) q_N^T$ , Weissauer asked if  $\sum_{T \in \mathcal{S}(\mathbb{Z})_{\geq 0}} \frac{1}{N} T A_F(T) q_N^T$  is a vector valued Siegel modular form [61]. Proposition 3.9 gives an affirmative answer to his question. That is nothing but the Siegel modular form  $\theta(F)$  of weight  $(k + p + 1, k + p - 1)$ .

Finally we explain how we reduce general weights  $\underline{k} = (k_1, k_2)$  of mod  $p$  Siegel modular form to more smaller weight by using Frobenius. Recall  $d_2 \circ Fr^* : R^1 f_* \mathcal{O}_{\mathcal{A}(p)} \rightarrow R^1 f_* \mathcal{O}_{\mathcal{A}}$  (see (3.3)). Combining this with the pull-back  $F_{\text{abs}}^* R^1 f_* \mathcal{O}_{\mathcal{A}(p)} = (R^1 f_* \mathcal{O}_{\mathcal{A}})^{(p)} \rightarrow R^1 f_* \mathcal{O}_{\mathcal{A}(p)}$  by  $F_{\text{abs}}$  on  $S_{N,p}$ , we have a canonical element in  $\text{Hom}_{\mathcal{O}_{S_{N,p}}}((R^1 f_* \mathcal{O}_{\mathcal{A}})^{(p)}, R^1 f_* \mathcal{O}_{\mathcal{A}})$ . By using principal polarization, we also have a canonical element in  $\text{Hom}_{\mathcal{O}_{S_{N,p}}}(\mathcal{E}^{\vee(p)}, \mathcal{E}^{\vee})$ . By using an identification  $\mathcal{E}^{\vee} \simeq \mathcal{E} \otimes \omega^{-1}$ , we have an element in  $\text{Hom}_{\mathcal{O}_{S_{N,p}}}(\mathcal{E}^{(p)}, \mathcal{E} \otimes \omega^{p-1})$ . Let us denote this element by  $\Phi$ . The map  $\Phi$  is naturally extended to the  $\mathcal{O}_{S_{N,p}}$ -linear map  $\Phi : \text{Sym}^k \mathcal{E}^{(p)} = (\text{Sym}^k \mathcal{E})^{(p)} \rightarrow \text{Sym}^k \mathcal{E} \otimes \omega^{k(p-1)}$ . Then the map  $\Phi$  induced a Hecke-equivariant morphism

$$H^0(\overline{S}_{N,p}, \text{Sym}^k \mathcal{E}^{(p)} \otimes \det^j \mathcal{E}) \hookrightarrow H^0(\overline{S}_{N,p}, \text{Sym}^k \mathcal{E} \otimes \det^{j+k(p-1)} \mathcal{E}) = M_{(k+j+k(p-1), j+k(p-1))}(\Gamma(N), \overline{\mathbb{F}}_p)$$

since  $S_{N,p}^h$  is dense open in  $S_{N,p}$ .

On the other hand, it is known that for any integer  $k \geq 0$ , as algebraic representations of  $SL_2/\overline{\mathbb{F}}_p$   $\rho_k = \text{Sym}^k St_2$  (and also their tensor product  $\rho_k \otimes \rho_{k'}$ ) can be obtained by successive extension of the following representation

$$\bigotimes_{i=1}^r \rho_{m_i}^{(p^{t_i})}, \quad m_i \in \{0, 1, \dots, p-1\}, \quad t_i \in \mathbb{Z}_{\geq 0}$$

(see [62]). Applying  $V$  and taking successive extension repeatedly, we have the following

**Theorem 3.15.** For any  $F \in M_{(k_1, k_2)}(\Gamma(N), \overline{\mathbb{F}}_p)$ , there exist an integer  $m$ ,  $0 \leq m \leq p-2$  and  $G \in M_{(k'_1, k'_2)}(\Gamma(N), \overline{\mathbb{F}}_p)$  with  $k'_1 \geq k'_2 \geq 1$  and  $p > k'_1 - k'_2$  such that  $\lambda_F(\ell^i) = \ell^{mi} \lambda_G(\ell^i)$  for  $\ell \nmid pN$ .

**3.4. Partial Hasse invariants on Ekedahl-Oort stratification.** In this subsection we will recall the Ekedahl-Oort stratification of  $S_{N,p}^*$  which consists of four strata and define partial Hasse invariants on the Zariski closures on each stratum. In [50] (see Claim-2) after Theorem (4.1)), Oort proved each stratum is quasi-affine by constructing a kind of Hasse invariant on each strata which is so called “Raynaud trick”.

To reduce the weight of a mod  $p$  Siegel modular form as in [15] for elliptic modular forms, we have to extend this invariants to the Zariski closure of the stratum in the Satake compactification of  $S_{N,p}$ .

As many people ([43], [25], [9], [14]) have studied an extension of this invariant to the Zariski closure of the strata, we also carry out such a kind of the argument but in this paper we apply a similar argument done in [14]. We will compute the local behavior of the partial Hasse invariants on each strata by using the corresponding display and it turns out to be holomorphic along the boundary locus in the Zariski closure.

Henceforth we will use the convention of [50] (see also [49] which would be a friendly reference for readers). The Ekedahl-Oort stratification has been studied thoroughly for Siegel modular variety of any degree  $g \geq 1$ . The combinatorial difficulty would come up when  $g$  tends to be large. However in our case, that is  $g = 2$ , we will not encounter this kind of trouble.

### 3.4.1. Ekedahl-Oort stratification.

**Definition 3.16.** A map  $\varphi : \{0, 1, 2\} \longrightarrow \{0, 1, 2\}$  is an elementary sequence if

- (1)
  - $\varphi(0) = 0$ ,
  - $\varphi(i-1) \leq \varphi(i) \leq \varphi(i-1) + 1$  for  $1 \leq i \leq 2$ .

For such  $\varphi$ , the number  $|\varphi| := \sum_{i=0}^2 \varphi(i)$  is called the dimension of  $\varphi$ . Let  $\Phi$  be the set of all elementary sequences. Then  $|\Phi| = 2^2 = 4$ .

The sequence  $\varphi$  so that  $\varphi(1) = \varphi(2) = 0$  is called the superspecial sequence and The sequence  $\varphi$  so that  $\varphi(i) = i$  for  $i = 1, 2$  is called the ordinary sequence.

- (2) We define the function  $f : \Phi \longrightarrow \mathbb{N}_{\geq 0}$  by

$$\begin{cases} f((1, 2)) = 2 \\ f = f(\varphi) \text{ such that } \varphi(f) = f = \varphi(f+1) \end{cases}$$

- (3) The number  $a = a(\varphi) := 2 - \varphi(2)$  is called  $a$ -number of  $\varphi$ .  
(4) For two elementary sequences  $\varphi_1, \varphi_2$ , we define an order so that  $\varphi_1 \prec \varphi_2$  if and only if  $\varphi_1(i) \leq \varphi_2(i)$ ,  $i = 1, 2$ .

For  $\varphi \in \Phi$ , it is determined by the values at 1 and 2. Therefore we represent  $\varphi$  as  $(\varphi(1), \varphi(2)) \in \{0, 1, 2\}^2$ . Then by definition, it is clear that

$$\Phi = \{(0, 0), (0, 1), (1, 1), (1, 2)\}.$$

We list up all  $f(\varphi)$  and  $a(\varphi)$  for each  $\varphi$  in convenience.

$\varphi$	(0, 0)	(0, 1)	(1, 1)	(1, 2)
$f = f(\varphi)$	0	0	1	2
$a = a(\varphi)$	2	1	1	0

TABLE 1.

**Definition 3.17.** A map  $\psi : \{0, 1, 2, 3, 4\} \longrightarrow \mathbb{Z}_{\geq 0}$  is a final sequence if

- $\psi(0) = 0$  and  $\psi(4) = 2$ ,



- $\varphi = (\psi(1), \psi(2))$  is an elementary sequence,
- $\psi(4-i) = \psi(i) + 2 - i$  for  $0 \leq i \leq 2$ .

Let  $\Psi$  be the set of all final sequences. Then  $|\Psi| = 2^2 = 4$ .

By definition there is the one-to-one correspondence between final sequences and elementary sequences. We represent a final sequence  $\psi$  as  $(\psi(1), \psi(2), \psi(3), \psi(4))$ . Let us list up all  $\psi$  for each  $\varphi$  in convenience.

$\varphi = (\varphi(1), \varphi(2))$	$(0, 0)$	$(0, 1)$	$(1, 1)$	$(1, 2)$
$\psi = (\psi(1), \psi(2), \psi(3), \psi(4))$	$(0, 0, 1, 2)$	$(0, 1, 1, 2)$	$(1, 1, 2, 2)$	$(1, 2, 2, 2)$

TABLE 2.

Let  $A = (A, \phi, \lambda)$  be an element of  $S_{N,p}(\overline{\mathbb{F}}_p)$  which represents a principal polarized abelian surface  $A$  over  $\overline{\mathbb{F}}_p$  with level structure  $\phi$ . Let us put  $G = G_A = A[p]$  and often drop the subscript  $A$  if the dependence is obvious from the context. Let  $F = F_G : G \rightarrow G^{(p)}$  be the relative Frobenius map and  $V = V_G : G^{(p)} \rightarrow G$  Verschiebung map. They satisfy

$$F \circ V = \text{id}_{G^{(p)}}, \quad V \circ F = \text{id}_G.$$

Further we have  $\text{Im} V = \text{Ker} F$  and  $\text{Im} F = \text{Ker} V$ . For any subgroup scheme  $H \subset G$  we write

$$(3.36) \quad V(H) := \text{Im}(V : H^{(p)} \rightarrow H), \quad F^{-1}(H) := F^{-1}(H^p).$$

The principal polarization  $\lambda : A \rightarrow A^\vee$  induces the isomorphism

$$(3.37) \quad \lambda : G_A \xrightarrow{\sim} G_{A^\vee} = G_A^D$$

where  $G_A^D$  is the Cartier dual of  $G_A$ . By using this we have a canonical non-degenerate pairing

$$\langle \cdot, \cdot \rangle : G \times G \xrightarrow{\sim} G \times G^D \rightarrow \mu_p.$$

For a subgroup scheme  $H \subset G$ , we define

$$H^\perp = \{g \in G \mid \langle g, h \rangle = 1 \text{ for any } h \in H\}$$

If  $H = \text{Im} V$ , then clearly  $H^\perp = H$ , hence  $H$  is maximal isotropic. By the polarization  $\lambda$ , we have the following relation:

$$(3.38) \quad \begin{array}{ccc} G & \xrightarrow{\lambda} & G^D \\ \downarrow F_G & & \uparrow V_{G^D} = (F_G)^D \\ G^{(p)} & \xrightarrow{\lambda^{(p)}} & (G^{(p)})^D = (G^D)^{(p)} \end{array} .$$

**Definition 3.18.** For  $G = A[p]$  as above, a final sequence for  $G$  is a filtration

$$0 = G_0 \subset G_1 \subset G_2 = H = \text{Im}(V) \subset G_3 \subset G_4 = G$$

such that there exists a final sequence  $\psi \in \Psi$  satisfying the conditions: for  $j \in \{0, 1, 2, 3, 4\}$

- $\text{rank}(G_j) = p^j$ ,
- $(G_j)^\perp = G_{4-j}$ ,

- $\text{Im}(V : G_j^{(p)} \longrightarrow G) = G_{\psi(j)},$
- $F^{-1}(G_i) = G_{2+i-\psi(j)}.$

We now study Ekedahl-Oort stratification of  $S_{N,p}$ . By Oort ([50], see also Theorem 2.7 of [49]), for each  $A = (A, \phi, \lambda) \in S_{N,p}(\overline{\mathbb{F}}_p)$ ,  $G = A[p]$  has a unique final sequence filtration corresponding to a unique  $\psi \in \Psi$ . Recall that the set  $\Psi$  is naturally identified with the set  $\Phi$  consisting of elementary sequences. Hence  $A$  gives rise to a unique elementary sequence  $ES(A) := \varphi$ . By using this facts, for each  $\varphi \in \Phi$ , one can define

$$S_\varphi := \{A \in S_{N,p}(\overline{\mathbb{F}}_p) \mid ES(A) = \varphi\}.$$

By definition,  $S_{(1,2)} = S_{N,p}^h$ . Let us consider the Zariski closure of  $S_\varphi$  in  $S_{N,p}$  which is denoted by  $\overline{S}_\varphi$ . Then we have

$$\overline{S}_\varphi = \coprod_{\varphi' \prec \varphi} S_{\varphi'}.$$

Clearly  $\overline{S}_{(1,2)} = S_{N,p}$ . It is well-known by [42] and [37] that

- (1)  $\overline{S}_{(1,1)}$  is a normal surface whose singularities happened exactly on all superspecial points are all  $A_1$ -type and they are locally isomorphic to  $t_{11}t_{22} - t_{12}^2 = 0$ ,
- (2)  $\overline{S}_{(0,1)}$  consists of smooth projective curves which are isomorphic to  $\mathbb{P}_{\overline{\mathbb{F}}_p}^1$ . The singularities of  $\overline{S}_{0,1}$  are exactly superspecial points and each components contains  $p^2 + 1$  superspecial points. Moreover, at each singular point there are  $p + 1$  irreducible components passing through and intersecting transversely.
- (3)  $\overline{S}_{(0,0)} = S_{(0,0)}$  is a finite set which consists of superspecial abelian surfaces over  $\overline{\mathbb{F}}_p$ .

We next consider the same kind of the stratification for the Satake compactification  $S_{N,p}^*$  of  $S_{N,p}$ .

**Definition 3.19.** Let  $X = (X, \phi, \lambda)$  be a semi abelian surface over  $\overline{\mathbb{F}}_p$  with a principal cubic structure  $\lambda$  and level structure  $\phi$  which is understood as an extension of group schemes

$$0 \longrightarrow (\mathbb{G}_m)^s \longrightarrow X \longrightarrow E \longrightarrow 0, \quad s \in \{1, 2\}$$

where if  $s = 1$ ,  $E = (E, \lambda_E, \phi_E)$  is an elliptic curve over  $\overline{\mathbb{F}}_p$  with level structure  $\phi_E : (\mathbb{Z}/N\mathbb{Z})_{\overline{\mathbb{F}}_p}^{\oplus 2} \xrightarrow{\sim} E[N]$  and principal polarization  $\lambda_E$  induced from  $\phi$  and  $\lambda$  respectively. We write  $X = ((\mathbb{G}_m)^s, E)$ . If  $s = 2$ ,  $E$  is a finite group scheme over  $\overline{\mathbb{F}}_p$  whose order is coprime to  $p$ . For such  $X = ((\mathbb{G}_m)^s, E)$  we associate an elementary sequence  $\varphi =: ES(X)$  as follows:

$$(\varphi(1), \varphi(2)) = \begin{cases} (1, 2) & \text{if } s = 2 \text{ or } s = 1 \text{ and } E \text{ is ordinary} \\ (1, 1) & \text{if } s = 1 \text{ and } E \text{ is supersingular.} \end{cases}$$

As was done for  $S_{N,p}$ , we define the stratum  $S_\varphi^*$  to be the set of semi-abelian surfaces over  $\overline{\mathbb{F}}_p$  with an elementary sequence  $\varphi$ . Let  $\overline{S}_\varphi^*$  be the Zariski closure of  $S_\varphi^*$  in  $S_{N,p}^*$ . Clearly,  $\overline{S}_{(1,2)}^* = S_{N,p}^*$  and it is easy to see that

$$\overline{S}_\varphi^* = \coprod_{\varphi' \prec \varphi} S_{\varphi'}^*.$$

Since  $\overline{S}_{\varphi'}$  with  $\varphi' \prec (0, 1)$  is closed,  $\overline{S}_\varphi^* = \overline{S}_\varphi$  and they never intersect with the boundary  $S_{N,p}^* \setminus S_{N,p}$ .

To define partial Hasse invariants which will be devoted to next subsection, it is better to work with canonical filtrations instead of final filtrations on  $G = A[p]$ . Let us explain what the canonical filtrations are (see (2.2) of [49]). Consider  $V^i(G) = \text{Im}(G^{(p^i)} \xrightarrow{V^{(p^{i-1})}} G^{(p^{i-1})} \xrightarrow{V^{(p^{i-2})}} \cdots \xrightarrow{V} G)$

which gives rise to a decreasing filtration on  $G$  (to be more precisely on the half  $V(G)$ ). Take all  $F^{-j}(V^i(G))$ ,  $j = 1, \dots$ , for each  $i$ . Then one has a filtration

$$\dots \subset F^{-n_1^{(i)}}(V^i(G)) \subset F^{-n_1^{(i)}}(V^i(G)) \subset \dots \subset F^{-n_{r_i}^{(i)}}(V^i(G)) \subset F^{-n_{r_i}^{(i)}}(V^{i-1}(G)) \subset \dots$$

for some integers  $0 < n_1 < \dots < n_{r_i}$ . Clearly  $F^{-n_j^{(i)}}(V^i(G)) \supset V(G)$ . This second filtration gives one between  $G$  and  $V(G)$ . We denote by  $\{V^i\}$  the first procedure and  $\{F^{-j}\}$  the second procedure symbolically. Repeating again first  $\{V^i\}$  and next  $\{F^{-j}\}$ , then at some step, the filtration between 0 and  $V(G)$  is stabilized by  $\{V^i\}$  and the filtration between  $V(G)$  and  $G$  is stabilized by  $\{F^{-j}\}$ . We write such a filtration as

$$0 = N_0 \subset \dots \subset N_r = V(G) \subset \dots \subset N_s = G$$

where  $r, s$  are non-negative integers. Then there exist functions

$$\rho : \{0, \dots, s\} \longrightarrow \mathbb{Z}_{\geq 0}, \quad v : \{0, \dots, s\} \longrightarrow \{0, \dots, r\}, \quad f : \{0, \dots, s\} \longrightarrow \{r, \dots, s\}$$

so that

$$(3.39) \quad \text{rank}(N_i) = p^{\rho(i)}, \quad V(N_i) = N_{v(i)}, \quad F^{-1}(N_i) = N_{f(i)}.$$

Since  $G$  has a polarization, the triple  $(\rho, v, f)$  is a symmetric canonical type in the sense of [50]. Clearly  $v$  and  $f$  are increasing functions. We write  $\Gamma_s := \{0, \dots, s-1\}$  and define  $\pi_G : \Gamma_s \longrightarrow \Gamma_s$  by

$$\pi_G(i) = \begin{cases} v(i) & \text{if } v(i+1) > v(i) \\ f(i) & \text{if } v(i+1) = v(i). \end{cases}$$

We write  $B_i = N_{i+1}/N_i$  for  $i \in \Gamma_s$ . By Lemma (2.4) of [49], the functions  $v$  and  $f$  are surjective and  $\pi_G$  is bijective. We denote by  $n(G)$  the order of  $\pi_G$ . Further,  $v$  and  $f$  satisfy the following properties:

- (1)  $v(i+1) > v(i) \iff f(i+1) = f(i)$  and in this case,  $V : B_i^{(p)} \longrightarrow B_{\pi(i)}$ ,
- (2)  $v(i+1) = v(i) \iff f(i+1) > f(i)$  and in this case,  $F : B_{\pi(i)} \longrightarrow B_i^{(p)}$ .

By the recipe given at (5.6) of [50], one can associate a triple  $\tau = (\rho, v, f)$  from each  $\varphi \in \Phi$ . We list up all datum as below (final column stands for a direct relation between the canonical filtration  $\{N_i\}_i$  and the final sequence  $\{G_i\}_i$  attached to each final sequence):

#### 3.4.2. Definition of partial Hasse invariants on Ekedahl-Oort stratification.

We are now in position to recall partial Hasse invariants  $H_\varphi$  on each strata  $S_\varphi^*$  for  $\varphi \in \Phi \setminus \{(0, 0)\}$ . Recall  $\omega := \det(\mathcal{E})$  where  $\mathcal{E}$  is the Hodge bundle on  $\overline{S}_{N,p} := \overline{S}_{K(N)} \otimes \mathbb{F}_p$  (see (2.18)). We use the same notation for  $\omega$  and  $\mathcal{E}$  on  $S_{N,p}$  or  $S_{N,p}^*$ . For each  $\varphi \in \Phi$ , by Serre duality we have  $(\omega^{-1}|_{\overline{S}_\varphi^*})^\vee \simeq \omega|_{\overline{S}_\varphi^*}$ .

For  $X = \mathcal{G} \times_{S_{N,p}} S_\varphi^*$ , put  $G = X[p]$ . We now apply the results in previous section by pulling back  $G$  to each strata.

In case  $\varphi = (1, 2)$ , we may put  $H_\varphi = H_{p-1}|_{S_{(1,2)}}$  where  $H_{p-1} \in H^0(S_{N,p}, \omega^{p-1})$  is the Hasse invariant.

In case  $\varphi = (1, 1)$ . From Table 3 we obtain

$$(3.40) \quad B_1^{(p^2)} \xleftarrow{\sim F^{(p)}} B_2^{(p)} \xrightarrow{\sim V} B_1, \quad B_0^{(p)} \xrightarrow{\sim V} B_0$$

$\varphi = (\varphi(1), \varphi(2))$	$(0, 0)$	$(0, 1)$	$(1, 1)$	$(1, 2)$
$(s, r)$	$(2, 1)$	$(4, 2)$	$(4, 2)$	$(2, 1)$
$(\rho(0), \dots, \rho(s))$	$(0, 2, 4)$	$(0, 1, 2, 3, 4)$	$(0, 1, 2, 3, 4)$	$(0, 2, 4)$
$(v(0), \dots, v(s))$	$(0, 0, 1)$	$(0, 0, 1, 1, 2)$	$(0, 1, 1, 2, 2)$	$(0, 1, 1)$
$(f(0), \dots, f(s))$	$(1, 2, 2)$	$(2, 3, 3, 4, 4)$	$(2, 2, 3, 3, 4)$	$(1, 1, 2)$
$(\pi_G(0), \dots, \pi_G(s-1))$	$(1, 0)$	$(2, 0, 3, 1)$	$(0, 2, 1, 3)$	$(0, 1)$
$n_\varphi := n_G$	2	4	2	1
canonical filtration	$N_i = G_{2i},$ $i \in \{0, 1, 2\}$	$N_i = G_i,$ $i \in \{0, \dots, 4\}$	$N_i = G_i,$ $i \in \{0, \dots, 4\}$	$N_i = G_{2i},$ $i \in \{0, 1, 2\}$

TABLE 3.

where  $B_i = N_{i+1}/N_i$ ,  $N_0 = 0$ ,  $N_1 = V^2(G)$ ,  $N_2 = \text{Ker}(F) = V(G)$ , and  $N_3 = \text{Ker}F^2 = F^{-1}(N_2)$ . The image or the inverse image should be understood under the convention (3.36). The filtration  $N_0 = 0 \subset N_1 \subset N_2 = V(G)$  induces the inclusions of tangent bundles:

$$0 \subset \mathfrak{t}_{N_1} \subset \mathfrak{t}_{N_2} = \mathfrak{t}_G = \mathfrak{t}_{X/S_\varphi^*}.$$

For  $i = 0, 1$  put  $\mathcal{L}_i = \det(\mathfrak{t}_{N_{i+1}}/\mathfrak{t}_{N_i})$ . Then one has  $\omega^{-1}|_{S_\varphi^*} = \det(\mathfrak{t}_{X/S_\varphi^*}) = \mathcal{L}_1 \otimes_{\mathcal{O}_{S_\varphi^*}} \mathcal{L}_2$ . Using this interpretation, for  $\varphi = (1, 1)$  we define

$$(3.41) \quad H_{(1,1)} = (V \circ F^{(p)^{-1}}) \otimes (V \circ V^{(p)}) : \underline{\omega}^{-p^2}|_{S_\varphi^*} \longrightarrow \underline{\omega}^{-1}|_{S_\varphi^*}.$$

It follows from this that  $H_{(1,1)}$  is regarded as a global non-where vanishing section of  $\underline{\omega}^{p^2-1}|_{S_\varphi^*}$ .

In case  $\varphi = (0, 1)$ . Similarly we obtain

$$(3.42) \quad \begin{array}{ccccccc} B_0^{(p^4)} & \xleftarrow{F^{(p^3)}} & B_2^{(p^3)} & \xleftarrow{F^{(p^2)}} & B_3^{(p^2)} & \xrightarrow{V^{(p)}} & B_1^{(p)} \xrightarrow{V} B_0, \\ B_1^{(p^4)} & \xrightarrow{V^{(p^3)}} & B_0^{(p^3)} & \xleftarrow{F^{(p^2)}} & B_2^{(p^2)} & \xleftarrow{F^{(p)}} & B_3^{(p)} \xrightarrow{V} B_1 \end{array}$$

where  $B_i = N_{i+1}/N_i$ ,  $N_0 = 0$ ,  $N_1 = VF^{-1}V(G)$ ,  $N_2 = V(G)$ , and  $N_3 = F^{-1}V(G)$ . The filtration  $N_0 = 0 \subset N_1 \subset N_2 = V(G)$  induces the inclusions of tangent bundles:

$$0 \subset \mathfrak{t}_{N_1} \subset \mathfrak{t}_{N_2} = \mathfrak{t}_G = \mathfrak{t}_{X/S_\varphi^*}.$$

For  $i = 0, 1$  put  $\mathcal{L}_i = \det(\mathfrak{t}_{N_{i+1}}/\mathfrak{t}_{N_i})$ . Then one has  $\omega^{-1}|_{S_\varphi^*} = \det(\mathfrak{t}_{X/S_\varphi^*}) = \mathcal{L}_1 \otimes_{\mathcal{O}_{S_\varphi^*}} \mathcal{L}_2$ . Then for  $\varphi = (0, 1)$  we define

$$(3.43) \quad H_{(0,1)} = (V \circ V^{(p)} \circ F^{(p^2)^{-1}} \circ F^{(p^3)^{-1}}) \otimes (V \circ F^{(p)^{-1}} \circ F^{(p^2)^{-1}} \circ V^{(p^3)}) : \underline{\omega}^{-p^4}|_{S_\varphi^*} \longrightarrow \underline{\omega}^{-1}|_{S_\varphi^*}.$$

It follows from this that  $H_\varphi$  is regarded as a global section of  $\underline{\omega}^{p^4-1}|_{S_\varphi^*}$ .

### 3.5. An extension of partial Hasse invariants.

According to [14] we compute the local behavior of partial Hasse invariants and check it can be naturally extended holomorphically on the Zariski closure  $\overline{S}_\varphi^*$  for each  $\varphi \neq (0,0)$ . The basic tools are the covariant Dieudonne theory for  $p$ -divisible groups and its local deformation. We refer [27], [14], and [50] for the notation.

For each  $X \in \overline{S}_{N,p}$  we denote by  $\mathbb{D} := \mathbb{D}(X^\vee[p])$  the Dieudonne module over  $\overline{\mathbb{F}}_p$  of  $X^\vee[p]$ . For any subgroup scheme  $H \subset X[p]$ , put  $\mathbb{D}(H) := \mathbb{D}(H^D)$  where  $H^D$  is the Cartier dual of  $H$ . Since the usual Dieudonne theory  $\mathbb{D}$  is contravariant, we have the covariant theory with the above definition.

For any  $\overline{\mathbb{F}}_p$ -module  $\mathcal{M}$  and a non-negative integer  $n$ , we define  $\mathcal{M}^{(p^n)}$  inductively by  $\mathcal{M}^{(p^n)} = \mathcal{M}^{(p^{n-1})} \otimes_{\overline{\mathbb{F}}_p, \phi} \overline{\mathbb{F}}_p$  where  $\phi$  is the Frobenius map. For an element  $m \in \mathcal{M}$  we put  $m^{(p)} = m \otimes 1$  and  $m^{(p^n)} = m^{(p^{n-1})} \otimes 1$ . In particular  $a^p m^{(p)} = m \otimes a^p = (am) \otimes 1 = (am)^{(p)}$  for  $a \in \overline{\mathbb{F}}_p$ .

The Frobenius map  $F$  and the Verschiebung  $V$  map on  $X[p]$  induce  $\overline{\mathbb{F}}_p$ -linear maps

$$\widehat{V} := F^\vee = F^D : \mathbb{D} \longrightarrow \mathbb{D}^{(p)}, \quad \widehat{F} := V^\vee = V^D : \mathbb{D}^{(p)} \longrightarrow \mathbb{D}.$$

The polarization induces a non-degenerate alternating pairing  $\langle *, * \rangle : \mathbb{D}^{(p)} \times \mathbb{D}^{(p)} \longrightarrow \overline{\mathbb{F}}_p^{(p)}$  and we have a relation

$$\langle \widehat{F}x, y^{(p)} \rangle = \langle x^{(p)}, \widehat{V}y \rangle^{(p)} \text{ for } x, y \in \mathbb{D}.$$

Then  $\widehat{V}$  is determined by  $\widehat{F}$  from this relation and vice versa.

By Table 4.3 of [31] for  $\varphi = (0,1)$  and Corollary in p.192 of [42] for  $\varphi = (0,0)$ , we have the represent matrices for  $\widehat{F}$  and  $\widehat{V}$  up to isomorphism for  $X \in S_\varphi^*$  (note that  $\widehat{V} = {}^t F$  and  $\widehat{F} = {}^t V$  where  $F$  are given in those references and  $V$  is obtained from  $F$ . Hence the role is interchanged due to the convention we have taken) :

$\varphi$	$\widehat{V}$	$\widehat{F}$
$(0,1)$	$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$
$(0,0)$	$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

TABLE 4.

We now study the local deformation of  $X \in S_\varphi^*$  for  $\varphi = (0,1)$  or  $(0,0)$ . We first consider the case  $\varphi = (0,1)$ . Let  $\mathcal{X}$  be the local deformation of  $X$  over the formal completion  $\widehat{S}_{N,p} = \mathrm{Spf} R$ ,  $R := \overline{\mathbb{F}}_p[[t_{11}, t_{12}, t_{22}]]$  of  $\overline{S}_{N,p}$  at  $X$ . We write  $\widehat{S}_{(1,1)}^*$  for the formal completion of  $\overline{S}_{(1,1)}^*$  at  $X$ . By (2.3) of

[27], the Frobenius map on the universal Dieudonné  $\mathbb{D}(\mathcal{X})$  over  $R$  is given by  $\widehat{V} = \begin{pmatrix} t_{12} & t_{22} & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

and left upper  $2 \times 2$  matrix is nothing but the Hasse matrix. On  $\overline{S}_{(1,1)}^*$  we must have  $\det \begin{pmatrix} t_{12} & t_{22} \\ 1 & 0 \end{pmatrix} = -t_{22} = 0$ . Therefore we have  $\overline{S}_{(1,1)}^* = \text{Spf} R_{(1,1)}$ ,  $R_{(1,1)} = \overline{\mathbb{F}}_p[[t_{11}, t_{12}]]$ . Put  $t = t_{12}$  for simplicity and then on  $\overline{S}_{(1,1)}^*$  the Frobenius and the Verschiebung on  $\mathbb{D}_1 := \mathbb{D}(\mathcal{X}) \otimes_R \overline{\mathbb{F}}_p[[t_{11}, t]]$  are given by

$$(3.44) \quad \widehat{V} = \begin{pmatrix} t & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \widehat{F} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & t & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

where  $\widehat{F}$  is computed from  $\widehat{V}$  via the alternative pairing. Applying first (3.40) for  $\mathcal{X}$  and taking the covariant Dieudonné module we have

$$(3.45) \quad \mathbb{D}(B_1)^{(p^2)} \xleftarrow{\widehat{V}^{(p)}} \mathbb{D}(B_2)^{(p)} \xrightarrow{\widehat{F}} \mathbb{D}(B_1), \quad \mathbb{D}(B_0)^{(p)} \xrightarrow{\widehat{F}} \mathbb{D}(B_0)$$

Note again that the role for  $F$  and  $V$  on  $\mathcal{X}$  is switched with  $\widehat{V}$  and  $\widehat{F}$  on the covariant Dieudonné module respectively. Let  $\{e_1, e_2, e_3, e_4\}$  be the basis of  $\mathbb{D}_1$  over  $R_{(1,1)}$  which satisfies  $\langle e_i, e_{i+2} \rangle = 1$  for  $i = 1, 2$  and  $\langle e_i, e_j \rangle$  for the remaining cases. We may assume that (3.44) has been given in this basis.

**Lemma 3.20.** *Keep the notation as above. The followings hold:*

- (1)  $\mathbb{D}(N_1) = \langle -e_2 + te_3 \rangle_{R_{(1,1)}}$ ,  $\mathbb{D}(N_2) = \langle e_2, e_3 \rangle_{R_{(1,1)}}$ , and  $\mathbb{D}(N_3) = \langle e_1 - te_4, e_2, e_3 \rangle_{R_{(1,1)}}$ ,
- (2)  $\mathbb{D}(B_0) = \langle -e_2 + te_3 \rangle_{R_{(1,1)}}$ ,  $\mathbb{D}(B_1) = \langle e_3 \rangle_{R_{(1,1)}}$ , and  $\mathbb{D}(B_2) = \langle e_1 - te_4 \rangle_{R_{(1,1)}}$ .

*Proof.* The first claim is done by direct computation without any difficulty and so details are omitted. Notice that  $\mathbb{D}$  is covariant. Therefore  $\mathbb{D}(B_i) = \mathbb{D}(N_{i+1})/\mathbb{D}(N_i)$ . Hence the second claim follows from (1).  $\square$

Using this Lemma we compute the local behavior of  $H_{(1,1)}$  along  $\widehat{S}_{(1,1)}^*$ .

**Proposition 3.21.** *There are two ways to extend the partial Hasse invariant  $H_{(1,1)} \in H^0(S_{(1,1)}^*, \omega^{p^2-1})$  to a section in  $H^0(\overline{S}_{(1,1)}^*, \omega^{p^2-1})$ . Let  $H_{(1,1)}^1$  and  $H_{(1,1)}^2$  be such sections. Then the zero locus (given by the reduced scheme structure) for each  $H_{(1,1)}^i$  is  $\overline{S}_{(0,1)}^* = \overline{S}_{(0,1)}$ . More precisely as a zero divisor,*

$$\text{div}_0(H_{(1,1)}^1) = \sum_{D \in \pi_0(\overline{S}_{(0,1)})} (p^2 + 2p - 1)D, \quad \text{div}_0(H_{(1,1)}^2) = \sum_{D \in \pi_0(\overline{S}_{(0,1)})} (2p)D$$

where  $\pi_0(\overline{S}_{(0,1)})$  stands for the set of all irreducible components of  $\overline{S}_{(0,1)}$ .

*Proof.* By (3.45) and Lemma 3.20 we may chase the image of a generator  $(e_1 - te_4)^{(p)} = e_1^{(p)} - t^p e_4^{(p)}$  of  $\mathbb{D}(B_2)^{(p)}$ . It follows that

$$\widehat{F}(e_1^{(p)} - t^p e_4^{(p)}) = -t^p e_3 \equiv -t^{p-1} e_2, \quad \widehat{V}^{(p)}(e_1^{(p)} - t^p e_4^{(p)}) = t^p e_1^{(p^2)} + e_2^{(p^2)} - t^p e_1^{(p^2)} = e_2^{(p^2)} \equiv t^{p^2} e_3^{(p^2)}.$$

On the other hand, for  $(-e_2 + te_3)^{(p^2)} = -e_2^{(p^2)} + t^{p^2} e_3^{(p^2)} \in \mathbb{D}(B_0)^{(p^2)}$ ,

$$\widehat{F} \circ \widehat{F}^{(p)}(-e_2^{(p^2)} + t^{p^2} e_3^{(p^2)}) = t^{p^2} \widehat{F}(-e_2^{(p)} + t^p e_3^{(p)}) = t^{p^2+p}(-e_2 + te_3).$$

According to (3.41) the partial Hasse invariant on  $\overline{S}_{(1,1)}^*$  is computed as

$$(\widehat{F} \circ (\widehat{V}^{(p)})^{-1}) \otimes (\widehat{F} \circ \widehat{F}^{(p)})(e_2^{(p^2)} \otimes (-e_2 + te_3)^{(p^2)}) = -t^{p^2+2p-1}(e_2 \otimes (-e_2 + te_3))$$



or

$$(\widehat{F} \circ (\widehat{V}^{(p)})^{-1}) \otimes (\widehat{F} \circ \widehat{F}^{(p)})(e_3^{(p^2)} \otimes (-e_2 + te_3)^{(p^2)}) = -t^{2p}(e_3 \otimes (-e_2 + te_3))$$

depending on the choice of the basis in  $\mathbb{D}(B_1)$ . The claim is now obvious from this.  $\square$

Next we consider the case  $\varphi = (0, 0)$ . Let  $\mathcal{X}$  be the local deformation of  $X$  over the formal completion  $\widehat{S}_{N,p} = \mathrm{Spf} R$ ,  $R := \overline{\mathbb{F}}_p[[t_{11}, t_{12}, t_{22}]]$  of  $\overline{S}_{N,p}$  at  $X$ . We write  $\widehat{S}_{(0,1)}^*$  for the formal completion of  $\overline{S}_{(0,1)}^*$  at  $X$ . By (2.3) of [27] again, the Frobenius map on the universal Dieudonné

$$\mathbb{D}(\mathcal{X}) \text{ over } R \text{ is given by } \widehat{V} = \begin{pmatrix} t_{11} & t_{12} & 1 & 0 \\ t_{12} & t_{22} & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ and left upper } 2 \times 2 \text{ matrix is nothing but the Hasse}$$

matrix. As explained in p.193 of [42] the local defining equation of  $\widehat{S}_{(0,1)}^* = \widehat{S}_{(0,1)}$  with a branch  $\zeta$  is given by

$$\widehat{S}_{(0,1)}^*[\zeta] : t_{11} = t, \ t_{12} = \zeta t, \ t_{22} = \zeta^2 t$$

where  $\zeta$  is any  $(p+1)$ -th root of  $-1$ . Therefore we have

$$\widehat{S}_{(0,1)}^* = \bigcup_{\zeta^{p+1}=-1} \widehat{S}_{(0,1)}^*[\zeta], \ \widehat{S}_{(0,1)}^*[\zeta] = \mathrm{Spf} R_{(0,1)}, \ R_{(0,1)} = \overline{\mathbb{F}}_p[[t]].$$

Then on  $\overline{S}_{(0,1)}^*$  the Frobenius and the Verschiebung on  $\mathbb{D}_0 := \mathbb{D}(\mathcal{X}) \otimes_R \overline{\mathbb{F}}_p[[t]]$  are given by

$$(3.46) \quad \widehat{V} = \begin{pmatrix} t & \zeta t & 1 & 0 \\ \zeta t & \zeta^2 t & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ \widehat{F} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & t & \zeta t \\ 0 & 0 & \zeta t & \zeta^2 t \end{pmatrix}.$$

Applying first (3.42) for  $\mathcal{X}$  and taking the covariant Dieudonné module we have

$$(3.47) \quad \begin{array}{ccccccc} \mathbb{D}(B_0)^{(p^4)} & \xleftarrow{\widehat{V}^{(p^3)}} & \mathbb{D}(B_2)^{(p^3)} & \xleftarrow{\widehat{V}^{(p^2)}} & \mathbb{D}(B_3)^{(p^2)} & \xrightarrow{\widehat{F}^{(p)}} & \mathbb{D}(B_1)^{(p)} \xrightarrow{\widehat{F}} \mathbb{D}(B_0), \\ \mathbb{D}(B_1)^{(p^4)} & \xrightarrow{\widehat{F}^{(p^3)}} & \mathbb{D}(B_0)^{(p^3)} & \xleftarrow{\widehat{V}^{(p^2)}} & \mathbb{D}(B_2)^{(p^2)} & \xleftarrow{\widehat{V}^{(p)}} & \mathbb{D}(B_3)^{(p)} \xrightarrow{\widehat{F}} \mathbb{D}(B_1) \end{array}$$

Let us introduce a basis  $\{e_1, e_2, e_3, e_4\}$  of  $\mathbb{D}_0 = \mathbb{D}(N_4)$  over  $R_{(0,1)}$  as in previous case. We may assume that (3.46) has been given in this basis.

**Lemma 3.22.** *Keep the notation as above. The followings hold:*

- (1)  $\mathbb{D}(N_1) = \langle e_1 - \zeta^{-1}e_2 \rangle_{R_{(0,1)}}$ ,  $\mathbb{D}(N_2) = \langle u_1, u_2 \rangle_{R_{(0,1)}}$ , and  $\mathbb{D}(N_3) = \langle -e_1 + \zeta e_2, u_1, u_2 \rangle_{R_{(0,1)}}$  where  $u_1 = -e_1 + t(e_3 + \zeta e_4)$ ,  $u_2 = -e_2 + t\zeta(e_3 + \zeta e_4)$ ,
- (2)  $\mathbb{D}(B_0) = \langle e_1 - \zeta^{-1}e_2 \rangle_{R_{(0,1)}}$ ,  $\mathbb{D}(B_1) = \langle u_1 \rangle_{R_{(0,1)}}$ ,  $\mathbb{D}(B_2) = \langle -e_1 + \zeta e_2 \rangle_{R_{(0,1)}}$ , and  $\mathbb{D}(B_3) = \langle e_3 \rangle_{R_{(0,1)}}$ .

*Proof.* This can be proved as Lemma 3.20 and so details are omitted.  $\square$

Using this Lemma we compute the order of  $H_{(0,1)}$  along  $\widehat{S}_{(0,1)}^*$ .

**Proposition 3.23.** *There is an extension of the partial Hasse invariant  $H_{(0,1)} \in H^0(S_{(0,1)}^*, \omega^{p^4-1})$  to a section in  $H^0(\overline{S}_{(0,1)}^*, \omega^{p^4-1})$ . The zero locus (given by the reduced scheme structure) of  $H_{(0,1)}$  is*

exactly given by  $\overline{S}_{(0,0)}^* = \overline{S}_{(0,0)}$ . More precisely as a zero divisor on each irreducible component  $D$  of  $\overline{S}_{(0,1)}$ ,

$$\operatorname{div}_0(H_{(0,1)}|_D) = \sum_{P \in \mathcal{S}_{(0,0)} \cap D} (p^4 - p^3 - p^2 + p)P.$$

*Proof.* As in previous case we apply (3.47) and Lemma 3.22. We first consider the first sequence of (3.47). We may chase the image of a basis  $e_3^{(p^2)}$  of  $\mathbb{D}(B_3)^{(p^2)}$ . It follows that  $\widehat{F}^{(p)}(e_3^{(p^2)}) = -e_1^{(p)} + t^p(e_3^{(p)} + \zeta^p e_4^{(p)})$  and

$$\widehat{F} \circ \widehat{F}^{(p)}(e_3^{(p^2)}) = t^p(u_1 + \zeta^p u_2) = t^p(u_1 - \zeta^{-1} u_2) = t^p(-e_1 + \zeta^{-1} e_2).$$

Similarly one has  $\widehat{V}^{(p^2)}(e_3^{(p^2)}) = e_1^{(p^3)}$  and

$$\widehat{V}^{(p^3)} \circ \widehat{V}^{(p^2)}(e_3^{(p^2)}) = t^{p^3}(e_1^{(p^4)} + \zeta^{p^3} e_2^{(p^4)}) = t^{p^3}(e_1^{(p^4)} - \zeta^{-1} e_2^{(p^4)}) = t^{p^3}(e_1 - \zeta^{-1} e_2)^{(p^4)}.$$

Therefore it follows

$$\widehat{F} \circ \widehat{F}^{(p)} \circ (\widehat{V}^{(p^3)} \circ \widehat{V}^{(p^2)})^{-1}((e_1 - \zeta^{-1} e_2)^{(p^4)}) = t^{-p^3+p}$$

Next we consider the second sequence of (3.47). It is easy to see that  $\widehat{F}(e_3^{(p)}) = u_1$ . In what follows we compare the images of  $u_1^{(p^4)}$  and  $e_3^{(p)}$  at the relay point  $\mathbb{D}(B_0)^{(p^2)}$ .

Note that  $e_3 \equiv \frac{1}{2}(e_3 - \zeta e_4)$  in  $\mathbb{D}(B_3)$ . Then  $e_3^{(p)} \equiv (\frac{1}{2}(e_3 - \zeta e_4))^{(p)} = \frac{1}{2}(e_3^{(p)} + \zeta^{-1} e_4^{(p)})$  in  $\mathbb{D}(B_3)^{(p)}$ . Therefore we have

$$\widehat{V}^{(p)}(e_3^{(p)}) = \widehat{V}^{(p)}(\frac{1}{2}(e_3^{(p)} + \zeta^{-1} e_4^{(p)})) = \frac{1}{2}(e_1^{(p)} + \zeta^{-1} e_2^{(p)}) = \frac{1}{2}(e_3 - \zeta e_4)^{(p)}$$

and

$$\widehat{V}^{(p^2)}(\frac{1}{2}(e_1^{(p)} + \zeta^{-1} e_2^{(p)})) = \frac{1}{2}\left\{t^{p^2}(e_1^{(p^3)} + \zeta^{p^2} e_2^{(p^3)}) + \zeta^{-1} t^{p^2} \zeta^{p^2} (e_1^{(p^3)} + \zeta^{p^2} e_2^{(p^3)})\right\} = t^{p^2}(e_1^{(p^3)} + \zeta e_2^{(p^3)}).$$

Since  $u_1^{(p^4)} = -e_1^{(p^4)} + t^{p^4}(e_3^{(p^4)} + \zeta^{p^4} e_4^{(p^4)}) = -e_1^{(p^4)} + t^{p^4}(e_3^{(p^4)} + \zeta e_4^{(p^4)})$ , we have

$$\widehat{F}^{p^3}(u_1^{p^4}) = t^{p^4}(-e_1^{(p^3)} - \zeta e_2^{(p^3)}) = t^{p^4-p^2}(\widehat{V}^{(p^2)} \circ \widehat{V}^{(p)})(e_3^{(p)}).$$

Put  $m = u_1 \otimes (e_1 - \zeta^{-1} e_2)$  and then  $m^{(p^4)} = u_1^{(p^4)} \otimes (e_1 - \zeta^{-1} e_2)^{(p^4)}$ . According to (3.41) the partial Hasse invariant on  $\widehat{\overline{S}}_{(0,1)}^*$  is computed as

$$(\widehat{F} \circ (\widehat{V}^{(p^2)} \circ \widehat{V}^{(p)})^{-1} \circ \widehat{F}^{p^3}) \otimes (\widehat{F} \circ \widehat{F}^{(p)} \circ (\widehat{V}^{(p^3)} \circ \widehat{V}^{(p^2)})^{-1})(m^{(p^4)}) = -t^{p^4-p^3-p^2+p} m$$

The claim is now obvious from this.  $\square$

By Koecher's principle, the Hasse invariant  $H_{p-1}$  is extended to a function on the Satake compactification  $S_{N,p}^*$  or a toroidal compactification  $\overline{S}_{N,p}$ . To end this section we give the following fact for  $H_{p-1}$  which is known well (cf [50]):

**Proposition 3.24.** *The zero locus of  $H_{p-1}$  is given by  $\overline{S}_{(1,1)}^*$  and the multiplicity of  $H_{p-1}$  along such locus is one.*

*Proof.* Let  $\mathcal{X}$  be the local deformation of  $X \in \overline{S}_{(1,1)}^*$  over the formal completion  $\widehat{\overline{S}}_{N,p} = \operatorname{Spf} R$ ,  $R := \widehat{\mathbb{F}}_p[[t_{11}, t_{12}, t_{22}]]$  of  $\overline{S}_{N,p}$  at  $X$ . Then  $\widehat{V} = F^\vee$  on  $\mathbb{D}(\mathcal{X})$  is given by the formal completion

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ t_{11} & t_{12} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Therefore  $H_{p-1}$  on  $\widehat{S}_{N,p}$  is given by the determinant of left upper  $2 \times 2$  matrix in  $\widehat{V}$ , that is  $t_{12}$ . Hence we have the claim.  $\square$

#### 4. WEIGHT REDUCTION

In this section we carry out weight reduction for mod  $p$  Siegel modular forms. Let  $\underline{k} = (k_1, k_2)$ ,  $k_1 \geq k_2 \geq 1$  satisfying  $p > k_1 - k_2 + 3$ . Recall  $\omega_{\underline{k}} = \text{Sym}^{k_1-k_2} \mathcal{E} \otimes_{\overline{S}_{N,p}} \omega^{k_2}$  where  $\mathcal{E}$  the Hodge bundle on a fixed toroidal compactification  $\overline{S}_{N,p}$  and  $\omega = \det \mathcal{E}$ . Let  $\mathcal{C} = \overline{S}_{N,p} - S_{N,p}$  be the boundary component. Put  $D_2 = \text{div}_0(H_{p-1})$  and  $D_1 = \text{div}_0(H_{(1,1)}^1)$ . Applying the construction in Section 3.5 to  $X = \mathcal{G} \times_{\overline{S}_{N,p}} D_1$ , one can extend  $H_{(0,1)}$  to  $D_1$  as a global section of  $\omega^{p^4-1}|_{D_1}$ . We denote it by  $H_{(0,1)}^{D_1}$  and put  $D_0 = \text{div}_0(H_{(0,1)}^{D_1})$ .

Then by Proposition 3.21, 3.23, and 3.24, we see that

$$D_2 = \overline{S}_{(1,1)}^*, \quad D_1^{\text{red}} = \overline{S}_{(0,1)}^* = \overline{S}_{(0,1)}, \quad D_0^{\text{red}} = \overline{S}_{(0,0)}^* = \overline{S}_{(0,0)} = S_{(0,0)}$$

where the superscript “red” stands for the reduced scheme structure. Let us consider the exact sequence

$$0 \longrightarrow \omega_{\underline{k}} \otimes w^{-(p-1)} \xrightarrow{\times H_{p-1}^{-1}} \omega_{\underline{k}} \longrightarrow \omega_{\underline{k}}|_{D_2} \longrightarrow 0.$$

This gives rise to a long exact sequence

$$(4.1) \quad \begin{array}{ccccccc} 0 & \longrightarrow & H^0(\overline{S}_{N,p}, \omega_{\underline{k}} \otimes w^{-(p-1)}) & \xrightarrow{\times H_{p-1}^{-1}} & H^0(\overline{S}_{N,p}, \omega_{\underline{k}}) & \longrightarrow & H^0(D_2, \omega_{\underline{k}}|_{D_2}) \\ & \longrightarrow & H^1(\overline{S}_{N,p}, \omega_{\underline{k}} \otimes w^{-(p-1)}) & \longrightarrow & H^1(\overline{S}_{N,p}, \omega_{\underline{k}}) & \longrightarrow & H^1(D_2, \omega_{\underline{k}}|_{D_2}) \\ & \longrightarrow & H^2(\overline{S}_{N,p}, \omega_{\underline{k}} \otimes w^{-(p-1)}) & \longrightarrow & H^2(\overline{S}_{N,p}, \omega_{\underline{k}}) & \longrightarrow & H^2(D_2, \omega_{\underline{k}}|_{D_2}) \\ & \longrightarrow & H^3(\overline{S}_{N,p}, \omega_{\underline{k}} \otimes w^{-(p-1)}) & \longrightarrow & H^3(\overline{S}_{N,p}, \omega_{\underline{k}}) & \longrightarrow & 0 \end{array}.$$

Note that  $\mathcal{E}^\vee \simeq \mathcal{E} \otimes \omega^{-1}$ . Then by Serre duality we have

$$\begin{aligned} H^3(\overline{S}_{N,p}, \omega_{\underline{k}} \otimes w^{-(p-1)}) &\simeq H^0(\overline{S}_{N,p}, \text{Sym}^{k_1-k_2} \mathcal{E}^\vee \otimes \omega^{p-1-k_2} \otimes_{\mathcal{O}_{\overline{S}_{N,p}}} \Omega_{\overline{S}_{N,p}}^3(-\mathcal{C})) \\ &\simeq H^0(\overline{S}_{N,p}, \text{Sym}^{k_1-k_2} \mathcal{E} \otimes \omega^{p+2-k_1}(-\mathcal{C})) = S_{(p+2-k_2, p+2-k_1)}(\Gamma(N), \overline{\mathbb{F}}_p) \end{aligned}$$

and it vanishes if  $p+2-k_2 < 0$ . Then plugging Theorem 2.3 into this long exact sequence we have the vanishing of cohomology:

**Proposition 4.1.** *Suppose that  $p > k_1 - k_2 + 3$ . The followings hold:*

- (1) *the restriction  $H^0(\overline{S}_{N,p}, \omega_{\underline{k}}) \longrightarrow H^0(D_2, \omega_{\underline{k}}|_{D_2})$  is surjective if  $k_2 > p+2$  or  $k_1 < p-1$ ,*
- (2)  *$H^1(D_2, \omega_{\underline{k}}|_{D_2}) = 0$  if  $k_2 > p+2$  or  $k_1 < p-1$ , and*
- (3)  *$H^2(D_2, \omega_{\underline{k}}|_{D_2})$  is embedded in  $S_{(k_1+p+2-2k_2, p+2-k_2)}(\Gamma(N), \overline{\mathbb{F}}_p)$  if  $k_2 > 3$  and further it vanishes if  $k_2 > p+2$ .*

*Proof.* For the second claim, we may use the duality

$$H^{3-i}(\overline{S}_{N,p}, \omega_{\underline{k}} \otimes w^{-(p-1)}) \simeq H^i(\overline{S}_{N,p}, \text{Sym}^{k_1-k_2} \mathcal{E} \otimes \omega^{p+2-k_1}(-\mathcal{C}))$$

for  $i = 1, 2$ . The remaining claims are clear.  $\square$

Similarly, by Proposition 3.21, there is an exact sequence

$$0 \longrightarrow \omega_{\underline{k}} \otimes w^{-(p^2-1)}|_{D_2} \xrightarrow{\times H_{(1,1)}^1} \omega_{\underline{k}}|_{D_2} \longrightarrow \omega_{\underline{k}}|_{D_1} \longrightarrow 0.$$

This gives rise to a long exact sequence

$$(4.2) \quad \begin{array}{ccccccc} 0 \longrightarrow & H^0(D_2, \omega_{\underline{k}} \otimes w^{-(p^2-1)}|_{D_2}) & \xrightarrow{\times H^1_{(1,1)}} & H^0(D_2, \omega_{\underline{k}}|_{D_2}) & \longrightarrow & H^0(D_1, \omega_{\underline{k}}|_{D_1}) \\ \longrightarrow & H^1(D_2, \omega_{\underline{k}} \otimes w^{-(p^2-1)}|_{D_2}) & \longrightarrow & H^1(D_2, \omega_{\underline{k}}|_{D_2}) & \longrightarrow & H^1(D_1, \omega_{\underline{k}}|_{D_1}) \\ \longrightarrow & H^2(D_2, \omega_{\underline{k}} \otimes w^{-(p^2-1)}|_{D_2}) & \longrightarrow & 0 & \longrightarrow & 0 \end{array}$$

Since  $D_2$  is a normal surface, it is Cohen-Macaulay. We then apply Serre duality to obtain

$$H^{2-i}(D_2, \omega_{\underline{k}} \otimes w^{-(p^2-1)}|_{D_2}) \simeq H^i(D_2, (\text{Sym}^{k_1-k_2} \mathcal{E} \otimes w^{p^2-1-k_1}(-\mathcal{C}))|_{D_2} \otimes \omega_{D_2})$$

where  $\omega_{D_2}$  is the dualizing sheaf of  $D_2$ . By adjunction formula we see that

$$\omega_{D_2} \simeq \omega^3|_{D_2} \otimes_{\mathcal{O}_{D_2}} \mathcal{O}(D_2)|_{D_2} = \omega^{p+2}|_{D_2}$$

since  $\mathcal{O}(D_2) = \mathcal{O}(\text{div}_0(H_{p-1})) = \omega^{p-1}$ . Plugging this into the above isomorphism we have

$$H^{2-i}(D_2, \omega_{\underline{k}} \otimes w^{-(p^2-1)}|_{D_2}) \simeq H^i(D_2, (\text{Sym}^{k_1-k_2} \mathcal{E} \otimes w^{p^2+p+1-k_1}(-\mathcal{C}))|_{D_2}).$$

Notice that Proposition 4.1 is still true even if we replace  $\omega_{\underline{k}}$  with  $\omega_{\underline{k}}(-\mathcal{C})$  by Theorem 2.3. Therefore we have the following.

**Proposition 4.2.** *Suppose  $p > k_1 - k_2 + 3$ .*

- (1) *the natural restriction  $H^0(D_2, \omega_{\underline{k}}|_{D_2}) \longrightarrow H^0(D_1, \omega_{\underline{k}}|_{D_1})$  is surjective if  $k_1 < p^2 + p - 2$  or  $k_2 > p^2 + 2$ .*
- (2) *any element of  $H^1(D_1, \omega_{\underline{k}}|_{D_1})$  is liftable to an element of  $S_{(p^2+p+1-k_2, p^2+p+1-k_1)}(\Gamma(N))$  if  $k_2 > p^2 + 2$  or  $k_1 < p^2 - 1$  and it vanishes if  $k_2 > p^2 + p + 1$ .*

*Proof.* By Proposition 4.1-(2),  $H^1(D_2, \omega_{\underline{k}} \otimes w^{-(p^2-1)}|_{D_2}) = 0$  if  $k_2 - (p^2 - 1) > p + 2$  or  $k_1 - (p^2 - 1) < p - 1$ , hence  $k_2 > p^2 + p + 1$  or  $k_1 < p^2 + p - 2$ . On the other hand,  $H^1(D_2, (\text{Sym}^{k_1-k_2} \mathcal{E} \otimes w^{p^2+p+1-k_1}(-\mathcal{C}))|_{D_2}) = 0$  if  $p^2 + p + 1 - k_1 > p + 2$  or  $p^2 + p + 1 - k_2 < p - 1$ , hence  $k_2 > p^2 + 2$  or  $k_1 < p^2 - 1$ . Summing up

$$H^1(D_2, \omega_{\underline{k}} \otimes w^{-(p^2-1)}|_{D_2}) = H^1(D_2, (\text{Sym}^{k_1-k_2} \mathcal{E} \otimes w^{p^2+p+1-k_1}(-\mathcal{C}))|_{D_2}) = 0$$

if  $k_1 < p^2 + p - 2$  or  $k_2 > p^2 + 2$ .

For the second claim, since  $k_2 > p + 2$  or  $k_1 < p - 1$ ,  $H^1(D_2, \omega_{\underline{k}}|_{D_2}) = 0$  by Proposition 4.1-(2). It follows from (4.2) that

$$H^1(D_1, \omega_{\underline{k}}|_{D_1}) \hookrightarrow H^2(D_2, \omega_{\underline{k}} \otimes w^{-(p^2-1)}|_{D_2}) \simeq H^0(D_2, (\text{Sym}^{k_1-k_2} \mathcal{E} \otimes w^{p^2+p+1-k_1}(-\mathcal{C}))|_{D_2}).$$

By Proposition 4.1-(1), the restriction map from  $H^0(\overline{S}_{N,p}, \text{Sym}^{k_1-k_2} \mathcal{E} \otimes w^{p^2+p+1-k_1}(-\mathcal{C}))$  to the right hand side of the above sequence is surjective provided if  $p^2 + p + 1 - k_1 > p + 2$  or  $p^2 + p + 1 - k_2 < p - 1$ .  $\square$

By Proposition 3.23, there is an exact sequence

$$0 \longrightarrow \omega_{\underline{k}} \otimes w^{-(p^4-1)}|_{D_1} \xrightarrow{\times H^1_{(0,1)}} \omega_{\underline{k}}|_{D_1} \longrightarrow \omega_{\underline{k}}|_{D_0} \longrightarrow 0.$$

This gives rise to a long exact sequence

$$(4.3) \quad \begin{array}{ccccccc} 0 \longrightarrow & H^0(D_1, \omega_{\underline{k}} \otimes w^{-(p^4-1)}|_{D_1}) & \xrightarrow{\times H^1_{(1,1)}} & H^0(D_1, \omega_{\underline{k}}|_{D_1}) & \longrightarrow & H^0(D_0, \omega_{\underline{k}}|_{D_0}) \\ \longrightarrow & H^1(D_1, \omega_{\underline{k}} \otimes w^{-(p^4-1)}|_{D_1}) & \longrightarrow & H^1(D_1, \omega_{\underline{k}}|_{D_1}) & \longrightarrow & 0. \end{array}$$

Unfortunately the curve  $D_1$  is not reduced, hence non-Cohen-Macaulay. However it is projective and we can apply more extensive duality theorem in [1]. The formally defined dualizing sheaf

$$\omega_{D_1} := \omega_{D_2}|_{D_1} \otimes_{\mathcal{O}_{D_1}} \mathcal{O}(D_1)|_{D_1} = \omega^{p^2+p+1}|_{D_1}$$

where the second equality follows from  $\mathcal{O}(D_1) = \mathcal{O}(\text{div}_0(H^1_{(1,1)})) = \omega^{p^2-1}|_{D_2}$  plays the same role in the duality

$$\begin{aligned} H^1(D_1, \omega_{\underline{k}} \otimes \omega^{-(p^4-1)}|_{D_1}) &\simeq H^0(D_1, (\text{Sym}^{k_1-k_2} \mathcal{E} \otimes w^{p^4-1-k_1}(-\mathcal{C}))|_{D_1} \otimes_{\mathcal{O}_{D_1}} \omega_{D_1}) \\ &\simeq H^0(D_1, (\text{Sym}^{k_1-k_2} \mathcal{E} \otimes w^{p^4+p^2+p-k_1})|_{D_1}) \end{aligned}$$

Then by Proposition 4.2-(1) we have

**Proposition 4.3.** *The natural map*

$$H^0(D_1, \omega_{\underline{k}}|_{D_1}) \longrightarrow H^0(D_0, \omega_{\underline{k}}|_{D_0})$$

*is surjective if  $k_2 > p^4 + p^2 + p$ .*

*Proof.* By Proposition 4.2 (2)  $H^1(D_1, \omega_{\underline{k}} \otimes \omega^{-(p^4-1)}|_{D_1}) = 0$  if  $k_2 - (p^4 - 1) > p^2 + p + 1$ . The claim now follows from the exact sequence (4.3).  $\square$

**Corollary 4.4.** *The restriction map*

$$H^0(\overline{S}_{N,p}, \omega_{\underline{k}}) \longrightarrow H^0(S_{(0,0)}, \omega_{\underline{k}}|_{S_{(0,0)}})$$

*is Hecke equivariant and surjective if  $k_2 > p^4 + p^2 + p$ .*

*Proof.* The first claim follows from the results of Ghitza ([22],[23]). Clearly the restriction map  $H^0(D_0, \omega_{\underline{k}}|_{D_0}) \longrightarrow H^0(S_{(0,0)}, \omega_{\underline{k}}|_{S_{(0,0)}})$  is surjective. Therefore the claim follows from Proposition 4.3.  $\square$

By Corollary 4.4 we will reduce the weight of mod  $p$  Siegel modular forms after the study of the images of theta operators.

**4.1. A non-vanishing criterion for theta operators.** In this section, we will study the image of theta operators and give a sufficient condition for which the image is not identically zero. A key is to observe the behavior when we restrict forms to superspecial locus  $S_{(0,0)}$  in  $S_{N,p}$ .

For any  $\underline{k} = (k_1, k_2)$ , let us recall the automorphic sheaf  $\omega_{\underline{k}}$  on  $\overline{S}_{N,p}$ . As Ghitza studied in [22], [23], we consider the space of superspecial forms:

$$SS_{\underline{k}} := H^0(S_{(0,0)}, \omega_{\underline{k}}|_{S_{(0,0)}}).$$

Let  $\mathbb{T}_N$  be the ring of Hecke operators  $T(\ell^i), i \geq 0$ ,  $\ell \nmid N$  over  $\mathbb{Z}$  if  $k_2 \geq 2$ ,  $T(\ell^i), i \geq 0$ ,  $\ell \nmid pN$  over  $\mathbb{Z}$  otherwise. For any  $\mathbb{T}_N$  module  $M$  over  $\overline{\mathbb{F}}_p$ , let us denote by  $HES(M)$  the set of the Hecke eigen systems for  $M$  (cf. [4]).

For any integer  $t > 0$ , put

$$M_t(\Gamma(N), \overline{\mathbb{F}}_p) := \bigoplus_{\substack{(k_1, k_2) \in \mathbb{Z}_{\geq 0}^2 \\ k_1 \geq k_2 \geq t}} M_{(k_1, k_2)}(\Gamma(N), \overline{\mathbb{F}}_p), \quad S_t(\Gamma(N), \overline{\mathbb{F}}_p) := \bigoplus_{\substack{(k_1, k_2) \in \mathbb{Z}_{\geq 0}^2 \\ k_1 \geq k_2 \geq t}} S_{(k_1, k_2)}(\Gamma(N), \overline{\mathbb{F}}_p),$$

and  $SS_t := \bigoplus_{\substack{(k_1, k_2) \in \mathbb{Z}_{\geq 0}^2 \\ k_1 \geq k_2 \geq t}} SS_{(k_1, k_2)}$ . Then in [23], Ghitza proved the following

**Theorem 4.5.** *For any integer  $t > 0$ ,*

$$HES(M_t(\Gamma(N), \overline{\mathbb{F}}_p)) = HES(S_t(\Gamma(N), \overline{\mathbb{F}}_p)) = HES(SS_t).$$

Actually the above statement is not written in [23], but it is obvious from the proof there. As a corollary we have the following:

**Corollary 4.6.** *Keep the notation as above. For any Hecke eigenform  $F$  in  $M_{(k_1, k_2)}(\Gamma(N))$ , there exists a Hecke eigen form  $G$  in  $S_{(k'_1, k'_2)}(\Gamma(N))$  for some weight  $(k'_1, k'_2)$  with  $k'_2 \geq k_2$  such that*

- (1) *the Hecke eigen systems of  $F$  and  $G$  are same,*
- (2) *the restriction of  $G$  to  $S_{(0,0)}$  is not identically zero.*

We are ready to study the non-vanishing of the images of mod  $p$  Siegel modular forms under theta operators.

**Theorem 4.7.** *Let  $F$  be an element in  $M_{(k,k)}(\Gamma(N))$ ,  $k \geq 2$ . Assume that  $F$  is not identically zero on  $S_{(0,0)}$ . The followings hold:*

- (1) *if  $p \nmid k$ , then  $\theta(F)$  is not identically zero and,*
- (2) *if  $p \nmid k(2k-1)$ , then  $\Theta(F)$  is not identically zero.*

*Proof.* By assumption, we may assume that  $\alpha := F|_{\{X\}}$  is non-zero for some  $X \in S_{(0,0)}$ . Let  $\mathcal{X}$  be the generic square zero deformation of  $X$  over  $\text{Spec } R$  (see the argument before Proposition 3.9).

By Corollary of Lemma 12 and 13 at p.192 of [42] we have the local behaviors of Hasse matrix and Hasse invariant

$$A(\mathcal{X}) \equiv \begin{pmatrix} t_{11} & t_{12} \\ t_{12} & t_{22} \end{pmatrix} \pmod{m_R^2}, \quad H(\mathcal{X}) \equiv t_{11}t_{22} - t_{12}^2 \pmod{m_R^3}.$$

Here we changed the normalization of  $A(\mathcal{X})$  by multiplying  $-1$ . By Proposition 3.5, the local expansion of  $\theta(F)$  at  $X$  is given by

$${}^t \begin{pmatrix} (t_{11}t_{22} - t_{12}^2)\nabla_{11}(F) + kt_{22}F \\ (t_{11}t_{22} - t_{12}^2)\nabla_{12}(F) + 2kt_{12}F \\ (t_{11}t_{22} - t_{12}^2)\nabla_{22}(F) + kt_{11}F \end{pmatrix} \equiv k\alpha \begin{pmatrix} t_{22} \\ 2t_{12} \\ t_{11} \end{pmatrix} \pmod{m_R^2}.$$

The first claim follows from this. Similarly, by Proposition 3.8, the local expansion of  $\Theta(F)$  at  $X$  is given by

$$\Theta(F)(\mathcal{X}) \equiv \frac{k(2k-1)\alpha}{9} \pmod{m_R}.$$

This proves the second claim. □

Next we study the vector valued case. Let us recall the notation in the proof of Proposition 3.12. Consider a local expansion of a non-zero element  $F = \sum_{i=0}^r F_i \delta_i$  in  $M_{(k_1, k_2)}(\Gamma(N), \overline{\mathbb{F}}_p)$  where we put  $r = k_1 - k_2 \geq 0$ . Assume that  $p > r$  (see Remark 3.13 for  $r = p-1, p-2$ ). We will consider two cases.

We first treat the case  $r = k_1 - k_2 = 1$ . In this case for  $F = F_0 \delta_0 + F_1 \delta_1$ , we have

$$(4.4) \quad \theta_2^{(k_1, k_2)}(F) = \det(A) \cdot \begin{pmatrix} \frac{1}{3}\nabla_{12}(F_0) - \frac{2}{3}\nabla_{11}(F_1) + \frac{2k_2-1}{3}(c_{11}F_1 - c_{12}F_0) \\ \frac{2}{3}\nabla_{22}(F_0) - \frac{1}{3}\nabla_{12}(F_1) + \frac{2k_2-1}{3}(c_{12}F_1 - c_{22}F_0) \end{pmatrix} \begin{pmatrix} \delta_0(\omega_1 \wedge \omega_2) \\ \delta_1(\omega_1 \wedge \omega_2) \end{pmatrix}.$$

**Theorem 4.8.** *Let  $F = F_0\delta_0 + F_1\delta_1$  be as above. Assume that  $F$  is not identically zero on  $S_{(0,0)}$ . Then if  $p \nmid (2k-1)$ ,  $\theta_2^{(k_1, k_2)}(F)$  is not identically zero.*

*Proof.* We use the notation in Theorem 4.7. Then the local expansion at  $X \in S_{(0,0)}$  is given by

$${}^t \left( \begin{array}{c} \frac{2k-1}{3}(t_{22}F_1|_X - t_{12}F_0|_X) \\ \frac{2k-1}{3}(t_{12}F_1|_X - t_{11}F_0|_X) \end{array} \right) \mod m_R^2.$$

The claim follows from this since either  $F_0|_X$  or  $F_1|_X$  is non-zero.  $\square$

Next we study the case  $k_1 - k_2 > 1$ , but we will be concerned with  $\theta_1^{(k_1, k_2)}$ , not  $\theta_2^{(k_1, k_2)}$ . By (3.35) and looking the construction of  $\theta_1$ , the coefficient of  $\tilde{\theta}_1^{(k_1, k_2)}(F)$  with respect to  $\delta_0 \otimes \omega_1^2 - \frac{1}{2}\delta_1 \otimes \omega_1\omega_2 + \delta_2 \otimes \omega_2^2$  is given by

$$(4.5) \quad (\nabla_{22}(F_0) - \frac{1}{2}\nabla_{12}(F_1) + \nabla_{11}(F_0)) - \frac{(k_1 - 3k_2)}{2}c_{22}F_0 - \frac{(k_1 - 3k_2)}{2}c_{12}F_1 - (k_1 - 3)c_{11}F_2$$

provided if  $c_{12} = c_{21}$ . Recall that  $\theta_1^{(k_1, k_2)}(F) = \det(A) \cdot \tilde{\theta}_1^{(k_1, k_2)}(F)$ . We are ready to discuss the nonvanishing of  $F$  under the theta operator  $\theta_1$  which decreases  $k_1 - k_2$  by  $-2$ .

**Theorem 4.9.** *Assume  $p > r := k_1 - k_2$ . Let  $F$  be as above and assume that  $F$  is not identically zero on  $S_{(0,0)}$ . If  $k_1 - 3k_2 \not\equiv 0 \mod p$ , then  $\theta_1^{(k_1, k_2)}(F)$  is not identically zero.*

*Proof.* By assumption we may assume  $F|_X \neq 0$  for some  $X \in S_{(0,0)}$ . By using the action of  $\rho_r\left(\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\right)$ ,  $a \in \overline{\mathbb{F}}_p$ , we may further assume that  $F_0|_X \neq 0$  without loss of generality. Let  $\mathcal{X}$  be the generic square zero deformation of  $X$  over  $\text{Spec } R$ . We proceed as the proof of Theorem 4.7. Then the local behavior of (4.5) is given by

$$-\frac{(k_1 - 3k_2)}{2}t_{11}F_0|_X - \frac{(k_1 - 3k_2)}{2}t_{12}F_1|_X - (k_1 - 3)t_{22}F_2|_X$$

which is clearly non-zero in  $R$ .  $\square$

**Remark 4.10.** *In the later purpose, we would be able to replace  $(k_1, k_2)$  with  $(k_1 + t(p+1), k_2 + t(p+1))$  for some non-negative integer  $t$  which is coprime to  $p$ . Then the assumption of Theorem 4.9 would be fulfilled.*

Next we construct a function on superspecial locus and reduce the weight of forms on the locus to be less than or equal to  $p+1$ .

For  $X \in S_{(0,0)}$ , let  $\mathcal{X}$  be the generic square zero deformation over  $R$ . Let  $m_X$  be the maximal ideal of  $\mathcal{O}_{S_{N,p}, X}$ . We may assume  $m_X/m_X^2 = m_R/m_R^2$ . Fix a local basis  $\omega_1, \omega_2$  of  $\mathcal{E}$  and let  $\omega_{i,X}$  be the restriction of  $\omega_i$  to  $X$ . As [15], for each  $X \in S_{(0,0)}$ , we consider the following local section of  $\omega^{p-1}|_{\{X\}} \otimes_{\overline{\mathbb{F}}_p} \text{Sym}^2(m_X/m_X^2) = \omega_{\mathcal{X}}^{p-1}|_{\{X\}} \otimes_{\overline{\mathbb{F}}_p} \text{Sym}^2(m_R/m_R^2)$ :

$$(\omega_{1,X} \wedge \omega_{2,X})^{p-1} \otimes \sum_{\substack{1 \leq i \leq j \leq 2 \\ 1 \leq k \leq l \leq 2}} \left( \frac{\partial}{\partial t_{ij}} \frac{\partial}{\partial t_{kl}} H_{p-1}(\mathcal{X}) \right) \Big|_X t_{ij} \otimes t_{kl}.$$

This section is independent of the choice of the local basis  $\omega_1, \omega_2$ .

It is easy to see that there is a canonical factor of  $\text{Sym}^2(m_X/m_X^2)$  which is isomorphic to  $\omega^2|_{\{X\}}$  and it is generated by  $t_{11} \otimes t_{22} - \frac{1}{4}t_{12} \otimes t_{12}$ . We define

$$(4.6) \quad B_X := \left\{ \left( \frac{\partial}{\partial t_{11}} \frac{\partial}{\partial t_{22}} - \frac{1}{4} \frac{\partial}{\partial t_{12}} \frac{\partial}{\partial t_{12}} \right) H_{p-1}(\mathcal{X}) \right\} \Big|_X (\omega_{1,X} \wedge \omega_{2,X})^{p+1}.$$



As mentioned above, this is independent of the choice of the local basis. Then  $B_X$  is a non-zero global section of  $\omega^{p+1}|_{\{X\}}$  since

$$\left\{ \left( \frac{\partial}{\partial t_{11}} \frac{\partial}{\partial t_{22}} - \frac{1}{4} \frac{\partial}{\partial t_{12}} \frac{\partial}{\partial t_{12}} \right) H_{p-1}(\mathcal{X}) \right\} \Big|_X = \left( \frac{\partial}{\partial t_{11}} \frac{\partial}{\partial t_{22}} - \frac{1}{4} \frac{\partial}{\partial t_{12}} \frac{\partial}{\partial t_{12}} \right) (t_{11}t_{22} - t_{12}^2) = 1 - \frac{1}{2} = \frac{1}{2}.$$

Hence the collection  $B := (B_X)_{X \in S_{(0,0)}}$  is an invertible function on  $S_{0,0}$  which is of weight  $p+1$ . In the sense of Ghitza,  $B$  is a superspecial form of weight  $p+1$  with level one.

**Proposition 4.11.** *The global section  $B$  induces an isomorphism*

$$SS_{(k_1, k_2)} \xrightarrow{\times B} SS_{(k_1+p+1, k_2+p+1)}.$$

Further  $T(\ell^i)(B \cdot F) = \ell^{2i} B \cdot T(\ell)(F)$  for all  $F \in SS_{(k_1, k_2)}$  and prime  $\ell \nmid pN$ .

*Proof.* As Lemma 7.4 of [15], the result follows from the compatibility of the Hecke action and the Kodaira-Spencer map with Section 3.2.1 of [23].  $\square$

By Theorem 3.15, Corollary 4.6, Theorem 4.9, and Theorem 4.11 with Remark 4.10 we have obtained the following

**Theorem 4.12.** *For any Hecke eigen form  $F \in M_{(k_1, k_2)}(\Gamma(N), \overline{\mathbb{F}}_p)$ , there exists a Hecke eigen form  $G \in S_{(k'_1, k'_2)}(\Gamma(N), \overline{\mathbb{F}}_p)$ ,  $k'_1 - k'_2 \in \{0, 1\}$  such that*

- (1)  $G$  is not identically zero on  $S_{(0,0)}$ ,
- (2) there exists an integer  $m \geq 0$  such that  $\lambda_F(\ell^i)(F) = \ell^{mi} \lambda_G(\ell^i)$  for any prime  $\ell \nmid pN$  and any integer  $i \geq 0$ .

We are now ready to prove the first main theorem:

**Theorem 4.13.** (Weight reduction theorem) *Suppose  $p \geq 5$ . For any Hecke eigen form  $F \in M_{(k_1, k_2)}(\Gamma(N), \overline{\mathbb{F}}_p)$ , there exists a Hecke eigen form  $G \in S_{(l_1, l_2)}(\Gamma(N), \overline{\mathbb{F}}_p)$  such that*

- (1)  $p > l_1 - l_2 + 3$  and  $l_2 \leq p^4 + p^2 + 2p + 1$ ,
- (2)  $G$  is not identically zero on  $S_{(0,0)}$ ,
- (3) there exists an integer  $0 \leq \alpha \leq p-2$  such that  $\lambda_F(\ell^i)(F) = \ell^{i\alpha} \lambda_G(\ell^i)$  for any prime  $\ell \nmid pN$  and any integer  $i \geq 0$ .

*Proof.* As already observed, one can reduce  $l_1 - l_2$  as small as possible. However we stop to the reduction for  $l_1 - l_2$  until the condition  $p > l_1 - l_2 + 3$  is fulfilled. If  $l_2 < p^4 + p^2 + 2p$ , we have nothing to prove. Otherwise by using superspecial modular form  $B$  of weight  $p+1$ , we may assume that there exists a Hecke eigen superspecial form  $G$  of weight  $(l_1, l_2)$  satisfying  $p > l_1 - l_2 + 3$  and  $1 \leq l_1 \leq p+1$ . Let  $i$  be the smallest positive integer so that  $i(p+1) + l_1 > p^4 + p^2 + p$ . We see that  $i = p^3 - p^2 + 2p$  if  $l_1 \neq 1$ ,  $i = p^3 - p^2 + 2p + 1$  if  $l_1 = 1$ . Then  $i(p+1) + l_1 \leq p^4 + p^2 + 2p + 1$ .  $\square$

*Proof.* (A proof of Theorem 1.2) By using the operator  $\theta_1^{(k_1, k_2)}$  one can reduce the difference  $k_1 - k_2$  with  $k_1 - k_2 - 2$ . Hence the parity is preserved under this operator. One can further reduce  $k_1 - k_2$  until it becomes 0 or 1 depending on the parity of  $k_1 - k_2$ . The weight  $k_2$  is reduced as we have seen in the proof of Theorem 4.13.  $\square$

A reason why we stop the reduction of  $l_1 - l_2$  is not only to work on the range of the weight so that vanishing theorem is applicable, but also to compare the local information of the mod  $p$  Galois representation associated to  $G$  in Theorem 4.13.

In this subsection we are concerning with the kernel of  $\theta$  and  $\Theta$ . We first introduce some notions.

**Definition 5.1.** Let  $F$  be an element in  $M_{(k,k)}(\Gamma(N), \overline{\mathbb{F}}_p)$ ,  $k \geq 2$  with the Fourier expansion  $F = \sum_{T \in \mathcal{S}(Z)_{\geq 0}} A_F(T) q_N^T$ .

- (1) We say  $F$  is a  $p$ -singular form if  $A(T) = 0$  unless  $p$  divides  $T$ .
- (2) We say  $F$  is a weak  $p$ -singular form if  $A(T) = 0$  unless  $p$  divides  $\det(T)$ .

Clearly, if  $F$  is  $p$ -singular, then  $F$  is weak  $p$ -singular. However the converse seems to be subtle. The following is a characterization of the  $p$ -singularity in terms of kernel of  $\theta$ .

**Theorem 5.2.** Let  $F$  be as above. Assume that  $p|k$ . Then  $F$  is a  $p$ -singular form if and only if  $\theta(F) = 0$ .

*Proof.* Put  $F = F_0(\omega_1 \wedge \omega_2)^k$  where  $F_0$  is a local section of  $\mathcal{O}_{S_{N,p}}$ . Recall the dual basis  $D_{ij}$  of  $d_{ij} = \langle \omega_i, \nabla \omega_j \rangle_{\text{dR}}$ . By Kodaira-Spencer,  $D_{ij}$ 's make up a basis of  $\text{Der}(\mathcal{O}_{S_{N,p}})$ . Then there exist local coordinates  $x_{11}, x_{12}, x_{22}$  of  $S_{N,p}$  so that  $(\frac{\partial}{\partial x_{11}}, \frac{\partial}{\partial x_{12}}, \frac{\partial}{\partial x_{22}}) = (D_{11}, D_{12}, D_{22})M$ ,  $M \in GL_3(\mathcal{O}_{S_{N,p}})$ .

By Proposition 3.5, the condition  $\theta(F) = 0$  is equivalent to  $D_{ij}(F_0) = 0$  for any  $1 \leq i \leq j \leq 2$ . This is also equivalent to  $\nabla(\frac{\partial}{\partial x_{ij}})(F_0) = 0$ ,  $1 \leq i \leq j \leq 2$  since  $M$  is invertible. Because  $S_{N,p}$  is smooth variety over  $\overline{\mathbb{F}}_p$ , this implies  $F_0 = G_0^p$  for some local section  $G_0$  of  $\mathcal{O}_{S_{N,p}}$ . Put  $k = pk'$ . Then we have  $F = (G_0(\omega_1 \wedge \omega_2)^{k'})^p$ . By gluing, there exists a  $G$  in  $M_{(k',k')}(\Gamma(N), \overline{\mathbb{F}}_p)$  such that  $F = G^p$ . Then  $F$  must have the following  $q$ -expansion  $F = \sum_{T \in \mathcal{S}(Z)_{\geq 0}} A_F(T) q^T$ . This implies  $F$  is a  $p$ -singular form.  $\square$

Obviously if  $\theta(F) = 0$ , then  $\Theta(F) = 0$ . However so far we do not know if the converse is always true.

## 6. THE $\theta$ -CYCLES

For a fixed integer  $r \geq 0$ , put  $M_r(N) := \bigoplus_{k \geq 0} M_{(r+k,k)}(\Gamma(N), \overline{\mathbb{F}}_p)$  and  $S_r(N) := \bigoplus_{k \geq 0} S_{(r+k,k)}(\Gamma(N), \overline{\mathbb{F}}_p)$ .

For  $F \in M_r(N)$ , we denote by  $\text{wt}(F)$  the weight of  $F$ . If  $F_1, F_2 \in M_r(N)$  have the same  $q$ -expansion with respect to the Mumford semi-abelian scheme, then  $\text{wt}(F_i) = (r + k_i, k_i)$ ,  $i = 1, 2$  satisfy  $k_1 - k_2 \equiv 0 \pmod{p-1}$ . This fact follows from the bigness of the monodromy of the Igusa tower (see the proof of Theorem 1 of [35]). In other words, if we write  $k_1 - k_2 = (p-1)m > 0$ , then we have  $F_1 = F_2 A^m$  in  $M_{(r+k_1, k_1)}(\Gamma(N), \overline{\mathbb{F}}_p)$ . The filtration  $w(F)$  of  $F \in M_r(N)$  is defined to be the smallest integer  $k$  for which there exists a form in  $M_{(r+k,k)}(N, \overline{\mathbb{F}}_p)$  which has the same  $q$ -expansion as  $F$  has. For any  $F \in M_r(N)$  there exists  $\tilde{F}$  such that  $\text{wt}(\tilde{F}) = w(F)$ ,  $\lambda_{\tilde{F}}(\ell^i) = \lambda_F(\ell^i)$  for any prime  $\ell \nmid pN$ ,  $i \geq 0$ , and  $\tilde{F}$  is not identically zero on the non-ordinary locus.

In this subsection, we make the following convention that

$$(6.1) \quad w(F) = w(G) \text{ if the Hecke eigensystems of } F \text{ and } G \text{ for } \mathbb{T}_N \text{ are same.}$$

We now study the filtration under the theta operators.

**Theorem 6.1.** (scalar valued case) Let  $F$  be an element in  $M_0(F)$  such that  $\tilde{F}$  is not identically zero on  $S_{(0,0)}$ . Then

- (1)  $w(\Theta(F)) \leq w(F) + p + 1$ . If  $p \nmid k(2k - 1)$ , then the equality holds and  $\Theta(F)$  is non-zero on  $S_{(0,0)}$ .
- (2)  $w(\Theta^{\frac{p+1}{2}}(F)) = w(\Theta(F))$ .
- (3) If  $p \nmid k$ , then  $w(\theta(F)) = w(F) + p - 1$ .

*Proof.* The inequality follows from the definition. By the proof of Theorem 4.7,  $\Theta(F)$  and  $\theta(F)$  are both non-zero at some point of  $S_{(0,0)}$ . The equality follows from this fact since the Hasse invariant is zero on such a point. The second claim is a consequence of Proposition 3.9-(1)-(a) with the convention (6.1).  $\square$

As in [36], [15] (the idea was due to J. Tate for elliptic modular forms), but we use a slight modification, the theta cycle of the scalar valued form  $F$  is defined by

$$\text{Cyc}(F) := (w(\Theta(F)), w(\Theta^2(F)), \dots, w(\Theta^{\frac{p-1}{2}}(F))).$$

Since  $w(\Theta(F)) = w(\Theta^{\frac{p+1}{2}}(F))$  under our convention (6.1), actually it makes up a “cycle”.

**Definition 6.2.** Let  $F$  be an element in  $M_0(N)$  with  $k \geq 2$ .

- (1) We say  $\Theta^i(F)$  is a low point of the first type (resp. the second type) if  $w(\Theta^{i-1}(F)) \equiv 0 \pmod p$  (resp.  $(2w(\Theta^{i-1}(F)) - 1) \equiv 0 \pmod p$ ). If  $F_i := \Theta^{c_i}(F)$  is a low point for some integer  $c_i > 0$ , then the number  $c_i - 1$  means one of times we add  $(p+1)$  to  $w(F)$ . We say  $c_i$  the low number of the low point  $\Theta^{c_i}(F)$ . We say  $c_i$  the low number for  $F_i$ . To avoid any confusion, we write  $c_i = c_i^{(1)}$  (resp.  $c_i = c_i^{(2)}$ ) if the low point is of the first type (resp. the second type).
- (2) We define the number  $b_i$  so that  $b_i(p-1) = w(\Theta^{c_i-1}F) + (p+1) - w(\Theta^{c_i}F)$  which means the amount falling the filtration at the low point  $F_i$  with the next application of  $\Theta$ . We say  $b_i$  the jumping number of the low point  $\Theta^{c_i}(F)$ . As  $c_i$ , we also write  $b_i = b_i^{(1)}$  or  $b_i = b_i^{(2)}$  according to the first type or the second type respectively.

Let  $\lambda_F(p^i)$  be the Hecke eigen-value of  $F$  for  $T(p^i)$ .

6.0.1. *non semi-ordinary case.* We first assume that  $\lambda_F(p^i) = 0$  for all  $i \geq 0$ . This is equivalent to  $\lambda_F(p^i) = 0$  for  $i = 1, 2$  by Proposition 3.3.35 at p.165 of [2]. We say  $F$  is non semi-ordinary at  $p$  when  $F$  has the eigenvalues as above. Otherwise we say  $F$  is semi-ordinary. By convention and definition, we have  $w(\Theta^{\frac{p-1}{2}}(F)) = w(F)$ . Let  $\{c_i\}_{i=1}^r$  (resp.  $\{b_i\}_{i=1}^r$ ) be the collection of all low numbers (jumping numbers) for  $F$ . We define  $F_i^{(m_i)}$ ,  $1 \leq i \leq r$  and  $m_i \in \{1, 2\}$  inductively so that  $m_{i+1} = j$  and  $F_{i+1}^{(m_{i+1})} = \Theta^{c_i^{(j)}}(F_i^{(m_i)})$ ,  $F_1 = F, m_1 = 1$ . Since the length of a theta cycle is  $\frac{p-1}{2}$ , one has  $\sum_i c_i = \frac{p-1}{2}$ . The total amount of the varying weights in a theta cycle is  $(p+1)\frac{(p-1)}{2}$ .

It follows from this that  $\sum_i b_i(p-1) = (p+1)\frac{(p-1)}{2}$ . Hence we have  $\sum_i b_i = \frac{p+1}{2}$ .

On the other hand,  $w(\Theta^{c_i^{(1)}-1}F_i^{(m_i)})(2w(\Theta^{c_i^{(2)}-1}F_i^{(m_i)}) - 1) \equiv 0 \pmod p$  for all  $i$ . From this we have

$$\begin{aligned} -b_i^{(1)} &\equiv 1 - w(F_{i+1}^{(1)}) \pmod p, \text{ or} \\ -2b_i^{(2)} &\equiv 3 - 2w(F_{i+1}^{(2)}) \pmod p. \end{aligned}$$

By definition we have  $w(\Theta^{c_{i+1}^{(j)}-1}F_{i+1}^{(j')}) = (c_{i+1}^{(j)} - 1)(p+1) + w(F_{i+1}^{(j')})$ ,  $j, j' \in \{1, 2\}$ . Then we also have

$$\begin{aligned} 0 &\equiv (c_{i+1}^{(1)} - 1) + w(F_{i+1}^{(j')}) \pmod{p}, \text{ or} \\ 1 &\equiv 2(c_{i+1}^{(2)} - 1) + 2w(F_{i+1}^{(j')}) \pmod{p}. \end{aligned}$$

for  $j' \in \{1, 2\}$ .

Putting these together, we have four cases for the consecutive low points and jumping numbers

$$(6.2) \quad \begin{aligned} \text{Case 1} \quad & b_i^{(1)} + c_{i+1}^{(1)} \equiv 0 \pmod{p}, \\ \text{Case 2} \quad & b_i^{(1)} + c_{i+1}^{(2)} \equiv \frac{p+3}{2} \pmod{p}, \\ \text{Case 3} \quad & b_i^{(2)} + c_{i+1}^{(1)} \equiv \frac{p-1}{2} \pmod{p}, \\ \text{Case 4} \quad & b_i^{(2)} + c_{i+1}^{(2)} \equiv 0 \pmod{p}. \end{aligned}$$

Put  $b_0 = c_{r+1} = 0$ . Since  $b_r + c_1 + \sum_{i=1}^{r-1} (b_i + c_{i+1}) = \sum_i (b_i + c_i) = p$ , we have the following for each case in (6.3):

$$(6.3) \quad \begin{aligned} \text{Case 1} \quad & r = 1 \text{ and } c_1^{(1)} = \frac{p-1}{2}, b_1^{(1)} = \frac{p+1}{2}, \\ \text{Case 2} \quad & r = 2 \text{ and } c_1^{(1)} + b_2^{(2)} = \frac{p-3}{2}, c_2^{(2)} + b_1^{(1)} = \frac{p+3}{2}, p \neq 5, \\ \text{Case 3} \quad & r = 2 \text{ and } c_2^{(1)} + b_1^{(2)} = \frac{p-1}{2}, c_1^{(2)} + b_2^{(1)} = \frac{p+1}{2}, \\ \text{Case 4} \quad & r = 1 \text{ and } c_1^{(2)} = \frac{p-1}{2}, b_1^{(2)} = \frac{p+1}{2}. \end{aligned}$$

In Case 2, since  $c_1^{(1)} + b_2^{(2)} \geq 2$ , this forces  $p$  to be greater than or equal to 7.

From now on we further assume that  $\tilde{F}$  is not identically zero on  $S_{(0,0)}$ . Put  $w(F) = ap + a'$ ,  $1 \leq a' \leq p$ ,  $a \in \mathbb{Z}_{\geq 0}$ .

Case 1.

Then  $w(\Theta^{c_1^{(1)}-1}F) = w(\Theta^{\frac{p-3}{2}}F) = ap + a' + \frac{(p-3)}{2}(p+1)$  by Theorem 6.1. The condition  $w(\Theta^{c_1^{(1)}-1}F) \equiv 0 \pmod{p}$  forces that  $a' \equiv \frac{(p+3)}{2} \pmod{p}$ . Note that if  $a' = \frac{(p+3)}{2}$ , then  $w(\theta^{c_1^{(1)}}F) = w(F) + \frac{(p-3)}{2}(p+1) + (p+1) - b_1^{(1)}(p-1) = w(F)$  which never contradicts with our setting. Summing up we have proved the following:

$$\text{Cyc}(F) = (k + (p+1), k + 2(p+1), \dots, k + \frac{(p-3)}{2}(p+1), k)$$

provided if  $w(F) \equiv \frac{(p+3)}{2} \pmod{p}$

Case 4.

This case is completely similar to Case 1. So we omit the details. We have

$$\text{Cyc}(F) = (k + (p+1), k + 2(p+1), \dots, k + \frac{(p-3)}{2}(p+1), k)$$

provided if  $w(F) \equiv 2 \pmod{p}$ .

Case 2.

Since  $w(\theta^{c_1^{(1)}}(F)) = ap + a' + (c_1^{(1)} - 1)(p + 1)$  has to be divisible by  $p$ , we have  $a' + (c_1^{(1)} - 1) \equiv 0 \pmod{p}$ . However  $1 \leq a' + (c_1^{(1)} - 1) \leq p + \frac{p-5}{2} < 2p$  (note that  $p > 5$  in this case). Hence we have  $a' + (c_1^{(1)} - 1) = p$ . Then we have

$$c_1^{(1)} = p + 1 - a', \quad c_2^{(2)} = a' - \frac{p+3}{2}, \quad b_1^{(1)} = p + 3 - a', \quad c_2^{(2)} = a' - \frac{p+5}{2}.$$

The filtrations are computed as follows:

$$w(\Theta^{c_1^{(1)}-1}(F)) = ap - a'p + p(p+1), \quad w(\Theta^{c_1^{(1)}}(F)) = w(\Theta^{c_1^{(1)}-1}F) - b_1^{(1)}(p-1) = ap + 4 - a',$$

and for  $F_1^{(1)} = \Theta^{c_1^{(1)}}(F)$ ,

$$w(\Theta^{c_2^{(2)}-1}(F_1^{(1)})) = ap + 4 - a' + (c_2^{(2)} - 1)(p+1) = ap + a'p - \frac{1}{2}(p^2 + 6p - 3).$$

However this contradicts with  $2w(\Theta^{c_2^{(2)}-1}(F_1^{(1)})) - 1 \equiv 0 \pmod{p}$ . Therefore this case does not occur.

Case 3.

We proceed as in Case 2. Since  $2w(\theta^{c_1^{(2)}}(F)) - 1 = 2ap + 2a' + 2(c_1^{(2)} - 1)(p+1) - 1$  is divisible by  $p$ , we must have  $a' + c_1^{(2)} = \frac{p+3}{2}$  or  $\frac{3p+3}{2}$ . The latter case does not occur since  $c_1^{(2)} + c_2^{(1)} = \frac{p-1}{2}$ . Therefore we have

$$c_1^{(2)} = \frac{p+3}{2} - a', \quad c_2^{(1)} = a' - 2, \quad b_1^{(2)} = \frac{p+3}{2} - a', \quad c_2^{(2)} = a' - 1.$$

This forces  $a'$  to satisfy

$$3 \leq a' \leq \frac{p+1}{2}.$$

The filtrations are computed as follows:

$$w(\Theta^{c_1^{(2)}-1}(F)) = ap - a'p + \frac{1}{2}(p+1)^2, \quad w(\Theta^{c_1^{(2)}}(F)) = w(\Theta^{c_1^{(2)}-1}F) - b_1^{(2)}(p-1) = ap + p + 3 - a',$$

and for  $F_1^{(2)} = \Theta^{c_1^{(2)}}(F)$ ,

$$w(\Theta^{c_2^{(1)}-1}(F_1^{(2)})) = ap + 4 - a' + (c_2^{(1)} - 1)(p+1) = ap + a'p - 2p.$$

Note that  $2w(\Theta^{c_1^{(2)}-1}(F)) - 1 \equiv 0 \pmod{p}$  and  $w(\Theta^{c_2^{(1)}-1}(F_1^{(2)})) \equiv 0 \pmod{p}$ . So there is no contradiction here. Put  $k_0 = a'$ . Then we have

$$\begin{aligned} \text{Cyc}(F) = & (k + (p+1), k + 2(p+1), \dots, k + (\frac{p+1}{2} - k_0)(p+1), \\ & k_1, k_1 + p + 1, \dots, k_1 + (k_0 - 3)(p+1)), \quad k_1 = k + p + 3 - 2k_0 \end{aligned}$$

provided if  $3 \leq k_0 \leq \frac{p+1}{2}$ .

Summing up we have proved

**Theorem 6.3.** *Let  $F$  be an element in  $M_0(N)$  with  $wt(F) \geq 2$  and  $\tilde{F}$  is not identically zero on  $S_{(0,0)}$ . Assume  $F$  is non semi-ordinary. Put  $w(F) = ap + k_0$ ,  $1 \leq k_0 \leq p$  and  $k_1 = k + p + 3 - 2k_0$ . Then*

$$\text{Cyc}(F) = \begin{cases} (k + (p+1), k + 2(p+1), \dots, k + \frac{(p-3)}{2}(p+1), k) & \text{if } w(F) \equiv 2 \pmod{p} \\ (k + (p+1), k + 2(p+1), \dots, k + \frac{(p-3)}{2}(p+1), k) & \text{if } w(F) \equiv \frac{p+3}{2} \pmod{p} \\ (k + (p+1), k + 2(p+1), \dots, k + (\frac{p+1}{2} - k_0)(p+1), \\ \quad k_1, k_1 + (p+1), \dots, k_1 + (k_0 - 3)(p+1), k) & \text{if } 3 \leq k_0 \leq \frac{p+1}{2} \\ (k + (p+1), k + 2(p+1), \dots, k + \frac{(p-3)}{2}(p+1), k) & \text{otherwise.} \end{cases}$$

Notice that the last case is just due to the convention (6.1).

6.0.2. *Semi-ordinary case.* Let  $F$  be an element in  $M_0(N)$  with  $wt(F) \geq 2$  and  $\tilde{F}$  is not identically zero on  $S_{(0,0)}$ . From now we assume  $F$  is semi-ordinary, hence  $\lambda_F(p) \neq 0$  or  $\lambda_F(p^2) \neq 0$ . Assume that  $k = w(F)$  satisfies both of  $k \not\equiv 0 \pmod{p}$  and  $k \not\equiv \frac{p+1}{2} \pmod{p}$ . Then  $w(\Theta(F)) = k + p + 1$ . Put  $k = ap + k_0$ ,  $1 \leq k_0 \leq p$  and  $k'_1 = k + p + 1 - 2k_0$ . Applying  $G = \Theta(F)$  to Theorem 6.3, we have

$$\text{Cyc}(F) = \begin{cases} (k + (p+1), k + 2(p+1), \dots, k + \frac{(p-1)}{2}(p+1)) & \text{if } w(F) \equiv 1 \pmod{p} \\ (k + (p+1), k + 2(p+1), \dots, k + \frac{(p-1)}{2}(p+1)) & \text{if } w(F) \equiv \frac{p+1}{2} \pmod{p} \\ (k + (p+1), k + 2(p+1), \dots, k + (\frac{p+1}{2} - k_0)(p+1), \\ k'_1 + (p+1), k'_1 + 2(p+1), \dots, k'_1 + (k_0 - 1)(p+1), k) & \text{if } 2 \leq k_0 \leq \frac{p-1}{2} \\ (k + (p+1), k + 2(p+1), \dots, k + \frac{(p-1)}{2}(p+1)) & \text{otherwise.} \end{cases}$$

The remaining cases are  $p|k$  or  $p|(2k-1)$ . We treat only  $k = p$  or  $k = \frac{p+1}{2}$ . When  $k = p$ , the possible values of  $w(\Theta(F)) = p + (p+1) - b_1(p-1)$ ,  $b_1 = 0, 1, 2, \dots$  are  $2p+1, p+2, 3$ . Then by Theorem 6.3 we have

$$\text{Cyc}(F) = \begin{cases} (2p+1, 2p+1 + (p+1), \dots, 2p+1 + \frac{(p-3)}{2}(p+1)) & \text{if } w(F) = 2p+1 \\ (p+2, p+2 + (p+1), \dots, p+2 + \frac{(p-3)}{2}(p+1)) & \text{if } w(F) = p+2 \\ (3, 3 + (p+1), \dots, 3 + \frac{(p-5)}{2}(p+1), p) & \text{if } w(F) = 3. \end{cases}$$

Similarly when  $k = \frac{p+1}{2}$ , the possible values of  $w(\Theta(F)) = \frac{p+1}{2} + (p+1) - b_1(p-1)$ ,  $b_1 = 0, 1, 2, \dots$  are  $\frac{3p+3}{2}, \frac{p+5}{2}$  if  $p > 7$  and  $9, 5, 1$  if  $p = 5$ . Then by Theorem 6.3 we have

$$\text{Cyc}(F) = \begin{cases} (\frac{3p+3}{2}, \frac{3p+3}{2} + (p+1), \dots, \frac{3p+3}{2} + \frac{(p-3)}{2}(p+1)) & \text{if } w(F) = \frac{3p+3}{2} \\ (\frac{p+5}{2}, \frac{p+5}{2} + (p+1), \dots, \frac{p+5}{2} + \frac{(p-3)}{2}(p+1)) & \text{if } w(F) = \frac{p+5}{2} \\ (1, 1 + (p+1), \dots, 1 + \frac{(p-3)}{2}(p+1)) & \text{if } p = 5 \text{ and } w(F) = 1. \end{cases}$$

In all lists of theta cycles, we have not used the assumption of begin semi-ordinary. As in Proposition 3.3 of [15], some of cycles might not occur according to whether  $F$  is  $p$ -singular or not.

6.0.3. *vector valued case.* In the following we study the theta cycles for vector valued forms  $F$  in  $M_1(N)$ . Assume  $F$  is of weight  $wt(F) = (k+1, k)$  and it is a Hecke eigenform. In this case we make use of the theta operator  $\theta_2^{k+1,k}$ . We drop the superscript and then simply denote it by  $\theta_2$ . We denote by  $w_2(F)$  the second component of the filtration  $w(F)$ . The following theta cycle looks like one in [15] and different from our scalar valued case:

$$\text{Cyc}(F) := (w_2(\theta_2(F)), w_2(\theta_2^2(F)), \dots, w_2(\theta_2^{p-1}(F))).$$

We still keep the convention (6.1). Let  $F(q) = \sum_T A_F(T) q_N^T$ ,  $A_F(T) \in St_2(\overline{\mathbb{F}}_p)$  be the Fourier expansion of  $F$ . By Theorem 4.8, we see that for any integer  $m \geq 0$ ,

$$\theta_2^{4m}(F)(q) = \sum_T \left( \frac{\det(T)}{18N^2} \right)^{2m} A_F(T) q_N^T.$$

Notice that  $w_2(\theta_2(F)) = w_2(\theta_2^p(F))$  under our convention (6.1) and Proposition 3.12-(2).

We use the notations of low point, low number, and jumping number in the case vector valued. Note that low point will happen only when  $p|(2k-1)$ . So the only second type can happen. By (the proof of) Theorem 4.8, if  $p \nmid (2k-1)$ , then  $w_2(F) = k + p$ .

We first assume that  $F$  is non semi-ordinary. Then by convention and definition, we have  $w_2(\theta_2^{p-1}(F)) = w_2(F)$ . Let  $\{c_i\}_{i=1}^r$  (resp.  $\{b_i\}_{i=1}^r$ ) be the collection of all low numbers (jumping numbers) for  $F$ . Note that in this case  $b_i$  is given by

$$b_i(p-1) = w_2(\theta_2^{c_i-1}F) + p - w_2(\theta_2^{c_i}F).$$

We define  $F_{i+1} = \theta_2^{c_i}(F_i)$ ,  $F_1 = F$  inductively. Since the length of a theta cycle is  $p-1$ , one has  $\sum_i c_i = p-1$ . The total amount of the varying weights (with respect to  $k$ ) in a theta cycle is  $p(p-1)$ . It follows from this that  $\sum_i b_i(p-1) = p(p-1)$ . Hence we have  $\sum_i b_i = p$ .

On the other hand,  $2w_2(\theta_2^{c_i-1}(F_i)) - 1 \equiv 0 \pmod{p}$  for all  $i$ . From this we have

$$-2b_i \equiv 1 - 2w_2(F_{i+1}) \pmod{p}.$$

By definition we have  $w(\theta_2^{c_{i+1}-1}(F_{i+1})) = (c_{i+1}-1)p + w(F_{i+1})$ . Then we also have

$$1 \equiv 2w_2(F_{i+1}) \pmod{p}.$$

Putting these together we have  $c_{i+1} \equiv \frac{p+1}{2} \pmod{p}$  and  $b_i \equiv 0 \pmod{p}$ . This forces us to have  $r=1$ ,  $b_1=p$ ,  $c_1=p-1$ . Put  $k = w_2(F)$ . Then  $2(p-2)p + 2k - 1$  has to be divided by  $p$ . Hence we have  $k \equiv \frac{p+1}{2} \pmod{p}$ . There is no difference between this case and other case because of the convention. Summing up we have

**Theorem 6.4.** *Let  $F$  be an element in  $M_1(N)$  and  $\tilde{F}$  is not identically zero on  $S_{(0,0)}$ . Assume  $F$  is non semi-ordinary. Then*

$$\text{Cyc}(F) = (k+p, k+2p, \dots, k+(p-2)p, k).$$

The semi-ordinary case is similar. Hence we have

**Theorem 6.5.** *Let  $F$  be an element in  $M_1(N)$  with  $k := w_2(F)$  and  $\tilde{F}$  is not identically zero on  $S_{(0,0)}$ . Assume  $F$  is semi-ordinary. Put  $k' = w_2(\theta_2(F))$ . Then*

$$\text{Cyc}(F) = (k'+p, k'+2p, \dots, k'+(p-1)p).$$

Further if  $k' = k+p$  if  $p \nmid (2k-1)$ .

The readers might have felt stress in that the weight in a cycle never fall except the last. To get more better cycle we might have to use the another theta operators. In the forthcoming paper [63] the author would suggests to study  $\tilde{\theta}_1^{(k_1, k_2)}(F) \otimes \theta(H_{p-1})$ .

## 7. GALOIS REPRESENTATIONS

Let  $F \in M_{k_1-k_2}(N)$  be a Hecke eigen cusp form with weight  $\text{wt}(F) = (k_1, k_2)$ ,  $k_1 \geq k_2 \geq 1$ . Assume that  $p \geq 5$  and  $p \nmid N$ . Thanks to the works [53],[55],[54],[46],[58],[59], by multiplying the Hasse invariant  $H_{p-1}$  if necessary, one can associate  $F$  with the mod  $p$  Galois representation

$$\overline{\rho}_{F,p} : G_{\mathbb{Q}} \longrightarrow \text{GSp}_4(\overline{\mathbb{F}}_p)$$

unramified outside  $N$  such that  $\det(I_4 - \overline{\rho}_{F,p}(\text{Frob}_{\ell})X) =$

$$1 - \lambda_F(\ell)X + (\lambda_F(\ell)^2 - \lambda_F(\ell^2) - \ell^{k_1+k_2-4}\chi_2(\ell))X^2 - \chi_2(\ell)\ell^{k_1+k_2-3}\lambda_F(\ell)X^3 + \chi_2(\ell)^2\ell^{2k_1+2k_2-6}X^4$$



for any  $\ell \nmid pN$ . Here  $\chi_2$  is the central character of  $F$ . We remark that we know a priori the above Galois representation takes the values in  $GL_4(\overline{\mathbb{F}}_p)$ . However by the classification of classical Siegel modular forms and the irreducibility results in [12] with [60], we may assume that the image is in  $GSp_4(\overline{\mathbb{F}}_p)$ . By the parity condition (2.7) and the comparison theorem, we have  $\nu \circ \overline{\rho}_{F,p}(c) = -1$  for any complex conjugation.

It is important to study the restriction of this representation to a smaller subgroup of  $G_{\mathbb{Q}}$  as the decomposition group  $D_p = \text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$  and the inertia group  $I_p = \text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p^{\text{ur}})$ . So let us now study the possible types of the images. Before proceeding, let us introduce some notation. Recall that  $GSp_4$  is defined by  $J$  in Section 1. Then we denote by  $B$  the Borel subgroup of  $GSp_4$  consisting of upper triangular matrices. The Siegel parabolic subgroup  $P$  and the Klingen parabolic subgroup  $Q$  are defined by

$$P = \left\{ \begin{pmatrix} A & B \\ 0_2 & D \end{pmatrix} \in GSp_4 \right\}, \quad Q = \left\{ \begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & 0 & 0 & * \end{pmatrix} \in GSp_4 \right\}$$

respectively. The endoscopic subgroup  $H$  of  $GSp_4$  is defined by

$$H = \left\{ \begin{pmatrix} a & 0 & 0 & b \\ 0 & x & y & 0 \\ 0 & z & w & 0 \\ c & 0 & 0 & d \end{pmatrix} \in GSp_4 \right\} \simeq (GL_2 \times GL_2) / \{(aI_2, aI_2) \mid a \in GL_1\}.$$

We now prove the following:

**Lemma 7.1.** *Let  $G$  be a group and  $L$  be a field with the characteristic away from 2. Let  $\sigma : G \rightarrow GSp_4(L)$  be a representation so that  $\iota \circ \sigma$  is reducible where  $\iota : GSp_4(L) \hookrightarrow GL_4(L)$  is the natural inclusion. Then up to conjugacy in  $GL_4(L)$ , the image  $\sigma$  takes the value in one of  $B(L), P(L), Q(L), H(L)$  depending on irreducible constituents of  $\iota \circ \sigma$ .*

*Proof.* Let  $V = L^{\oplus 4}$  be the representation space of  $\iota \circ \sigma$  and  $\langle *, * \rangle$  be the alternating form on  $V$  defined by  $J$ . Since  $V$  is reducible as a  $GL_4(L)$ -module, there exists the non-trivial minimal  $GL_4(L)$ -subspace  $W$ . Assume that  $\dim_L W = 1$ . Fix a non-zero element  $e_1$  in  $W$ . Consider the map  $V \rightarrow L, v \mapsto \langle e_1, v \rangle$  which is clearly surjective because of the non-degeneracy of the pairing. Take an element  $e_4$  so that  $\langle e_1, e_4 \rangle = 1$  and consider the map  $V \rightarrow L \oplus L, v \mapsto (\langle e_1, v \rangle, \langle e_4, v \rangle)$  which is also surjective. Take a basis  $e_2, e_3$  of the kernel of this map. Since the dimension of the maximal totally isotropic subspace of  $V$  is 2, one must have  $\langle e_2, e_3 \rangle \neq 0$ . So we may assume that  $\langle e_2, e_3 \rangle = 1$ . Then we have  $J = (\langle e_i, e_j \rangle)_{1 \leq i, j \leq 4}$ . In this case the image is in  $Q(L)$ . Further there exists a  $GL_4(L)$ -subspace  $W'$  containing  $W$  as a proper submodule, then the image is included in  $B(L)$ .

We omit the proof for remaining cases, but gives the relation between  $W$  and the image. If  $\dim_L W = 2$  and  $W$  is totally isotropic, then the image is in  $P(L)$ . If  $\dim_L W = 2$  and  $W$  is not totally isotropic, then the image is in  $H(L)$ . If  $\dim_L W = 3$ , this case is included in the first case by duality.  $\square$

Let  $\bar{\rho} : G_{\mathbb{Q}} \longrightarrow GSp_4(\overline{\mathbb{F}}_p)$  be a continuous representation. We are concerning with the types of  $\bar{\rho}|_{I_p}$ .

**Proposition 7.2.** *Up to conjugacy, the representation  $\bar{\rho}_p|_{I_p}$  is equivalent to one of the following four cases:*

$$\begin{aligned}
& \text{(Borel type)} \quad \begin{pmatrix} \bar{\chi}_p^a & * & * & * \\ 0 & \bar{\chi}_p^b & * & * \\ 0 & 0 & \bar{\chi}_p^c & * \\ 0 & 0 & 0 & \bar{\chi}_p^d \end{pmatrix}, \quad 0 \leq a, b, c, d \leq p-2, \quad a+d \equiv b+c \pmod{p}, \\
& \text{(Klingen type)} \quad \begin{pmatrix} \bar{\chi}_p^a & * & * & * \\ 0 & \phi_2^{b+cp} & 0 & * \\ 0 & 0 & \phi_2^{pb+c} & * \\ 0 & 0 & 0 & \bar{\chi}_p^d \end{pmatrix}, \quad \begin{matrix} 0 \leq a, d \leq p-2, \\ 0 \leq b < c \leq p-1, \\ a+d \equiv b+c \pmod{p} \end{matrix}, \\
& \text{(Siegel type)} \quad \begin{pmatrix} \phi_2^{a+bp} & 0 & * & * \\ 0 & \phi_2^{pa+b} & * & * \\ 0 & 0 & \bar{\chi}_p^k \phi_2^{-(a+bp)} & 0 \\ 0 & 0 & 0 & \bar{\chi}_p^k \phi_2^{-(pa+b)} \end{pmatrix}, \quad \begin{matrix} 0 \leq a < b \leq p-1, \\ (\nu \circ \bar{\rho})|_{I_p} = \bar{\chi}_p^k, \quad 0 \leq k \leq p-2 \end{matrix}, \\
& \text{(Endoscopic type)} \quad \begin{pmatrix} \phi_2^{a+bp} & 0 & 0 & 0 \\ 0 & \phi_2^{c+pd} & 0 & 0 \\ 0 & 0 & \phi_2^{pc+d} & 0 \\ 0 & 0 & 0 & \phi_2^{pa+b} \end{pmatrix}, \quad \begin{matrix} 0 \leq a < b \leq p-1, \\ 0 \leq c < d \leq p-1, \\ a+b \equiv c+d \pmod{p} \end{matrix}, \\
& \text{(Level 4 type)} \quad \text{diag}(\phi_4^a, \phi_4^{ap}, \phi_4^{ap^2}, \phi_4^{ap^3}), \quad \begin{matrix} a = a_0 + a_1p + a_2p^2 + a_3p^3, \quad 0 \leq a_i \leq p-1, \\ a \not\equiv 0 \pmod{p^2+1}, \quad a \equiv 0 \pmod{p+1}. \end{matrix}
\end{aligned}$$

where  $\phi_n$  stands for the fundamental character of level  $n$  (see p.213-214 of [16] for the definition). Further these five images takes the value in  $GSp_4(\overline{\mathbb{F}}_p)$ .

*Proof.* We have only to prove when  $\bar{\rho}|_{D_p}$  is irreducible. Remaining cases are easy to check. On the images, we may apply Lemma 7.1.

By standard method as in p.214 of [16], we may assume that in  $GL_4$ ,

$$\bar{\rho}|_{D_p} \sim \text{diag}(\phi_4^a, \phi_4^{ap}, \phi_4^{ap^2}, \phi_4^{ap^3}), \quad a = a_0 + a_1p + a_2p^2 + a_3p^3, \quad 0 \leq a_i \leq p-1$$

where  $\phi_4$  is the fundamental character of level 4 and  $a$  can not be divisible by  $p^2+1$ . Note that  $\phi_4$  is of order  $p^4-1$ . Since we know a priori  $\bar{\rho}|_{D_p} \subset GSp_4(\overline{\mathbb{F}}_p)$ , at least one of the followings should occur:

$$\phi_4^{a+ap} = \phi_4^{ap^2+ap^3}, \quad \phi_4^{a+ap^2} = \phi_4^{ap+ap^3}, \quad \phi_4^{a+ap^3} = \phi_4^{ap+ap^2}.$$

Further since  $(\nu \circ \bar{\rho})|_{I_p}$  is a power of  $\chi_p$ , so are the above ones. Then in the first case,  $p^4-1$  should divide  $a(p^2-1)$ . Hence  $a$  is divisible by  $p^2+1$  which gives a contradiction. Similarly we have a contradiction in the third case. Therefore only second case can occur and in this case we must have  $a \equiv 0 \pmod{p+1}$ .  $\square$

## 8. APPENDIX A

In this section we give an explicit form of Pieri's decomposition. Let  $R$  be a  $\overline{\mathbb{F}}_p$ -algebra. Put  $St_2(R) = Re_1 \oplus Re_2$  and  $GL_2(R)$  acts on  $St_2$  by

$$ge_1 = ae_1 + ce_2, \quad ge_2 = be_1 + de_2, \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

For positive integer  $n$ , let  $V(n) = \text{Sym}^n St_2(R)$  be the  $n$ -th symmetric representation of  $GL_2(R)$ . Put  $V(n, m) = V(n) \otimes \det^n(St_2(R))$  for integers  $m \geq 0$ .

We are concerning with an explicit decomposition of  $V(n) \otimes V(2)$ . Consider the basis  $u_i = e_1^i e_2^{n-i}$ ,  $i = 0, \dots, n$  (resp.  $v_2 = e_1^2$ ,  $v_1 = e_1 e_2$ ,  $v_0 = e_2^2$ ) of  $V(n)$  (resp.  $V(2)$ ). We define the operators  $E, F$  on  $V(n)$  by  $Eu_i = iu_{i-1}$ ,  $Fu_i = (n-i)u_{i+1}$ . We also define the same operators on  $V(n) \otimes_R V(n')$  by Leibniz rule  $E(u_i \otimes u_j) = Eu_i \otimes u_j + u_i \otimes Eu_j$ ,  $F(u_i \otimes u_j) = Fu_i \otimes u_j + u_i \otimes Fu_j$ .

We first assume that  $n+3 \leq p$ . We now try to decompose  $V(n) \otimes V(2)$ . Put

$$w_0 = u_n \otimes v_2, \quad w_1 = u_n \otimes v_1 - u_{n-1} \otimes v_2, \quad w_2 = u_n \otimes v_0 - 2u_{n-1} \otimes v_1 + u_{n-2} \otimes v_2$$

Since  $Fw_i = 0$ ,  $i = 0, 1, 2$ , we expect that these would be highest weight vectors. Put

$$W_0(n) = \langle f_i^{(0)} := \frac{1}{(n+2)_i} E^i w_0, \quad 0 \leq i \leq n+2 \rangle_R, \quad W_1(n) = \langle f_i^{(1)} := \frac{1}{(n)_i} E^i w_1, \quad 0 \leq i \leq n \rangle_R,$$

and  $W_2(n) = \langle f_i^{(2)} := \frac{1}{(n-2)_i} E^i w_2, \quad 0 \leq i \leq n-2 \rangle_R$  where  $(*)_i$  stands for the Pochhammer symbol and we set  $(*)_0 = 1$  and  $(m)_m = 1$ .

By direct calculation we have the following:

**Proposition 8.1.** *For  $j = 0, 1, 2$ , as  $GL_2(R)$ -modules,*

$$V(n+2-2j, j) \xrightarrow{\sim} W_j(n), \quad u_i \mapsto f_i^{(j)}.$$

When  $n = p-1$  or  $p-2$ , notice that the denominators of the coefficients appearing in  $f_i^{(2)} := \frac{1}{(n-2)_i} E^i w_2$  are not divisible by  $p$ . Hence only  $W_2(n)$  still make sense and so does the isomorphism  $V(n-2, 2) \xrightarrow{\sim} W_2(n)$ . Clearly  $W_2(n)$  gives a splitting of the surjection  $V(n) \otimes V(2) \longrightarrow V(n-2, 2)$ .

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